Polyhedral study of a new formulation for the Rural Postman Problem

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Abstract

In this work we study a special case of the well-known Rural Postman Problem (RPP) with the characteristic of being defined in a graph that has a non-required edge parallel to each required one. We formulate the problem with three binary variables associated with the traversal of a required edge and its parallel non-required one. Although some variables are superfluous, in addition to being interesting by itself, this model is attractive because it is the special case for $K = 1$ of the RPP with K vehicles (K-RPP) and its polyhedral study is necessary for that of the general case.

We study the polyhedron defined by the convex hull of the set of feasible solutions of the problem and show that several wide families of inequalities induce facet of this polyhedron under mild conditions.

Keywords: Rural Postman Problem, facet, polyhedron.

1 Introduction

Arc routing problems (ARPs) are those problems in which one or more vehicles have to meet the demand of some customers (modeled as edges or arcs of a graph) so that a given objective is optimized. Customers can be city streets, highways, lines that define contours of figures, etc., whose cleaning must be carried out, their snow must be removed, or the contour cut, for example. The objective can be to minimize the total distance traveled, balance the duration of the routes, or maximize the benefit obtained by serving customers, for example. The reader can consult the references [11], [8], [5], [16], and [4] to find information on models, applications and solution procedures for ARPs.

In this paper we deal with the *Rural Postman Problem* (RPP). The RPP is a generalization of the well-known *Chinese Postman Problem* ([13]), which was introduced by Orloff [18] and can be defined as follows. Given an undirected graph $G = (V, E)$, and a set of "required" edges E_R (representing those that must be serviced), the RPP is to find a minimum cost closed walk traversing each edge in E_R at least once. If the graph induced by E_R is connected, the RPP can be solved in polynomial time, otherwise, it is NP-hard ([15]).

When the number of required edges is large, a single vehicle may not be sufficient to service all customers and several vehicles are needed. These multi-vehicle problems are much more difficult to solve and require conditions to balance the lengths of the routes of the different vehicles. A widely used condition is to limit the length of the route of each vehicle by a certain value (determined by the capacity or autonomy of the vehicles, by the working hours of the drivers, etc.). This problem, *the Length Constrained K–vehicles Rural Postman Problem* (LC K-RPP) has been studied in [2] and [3] and can be defined as follows.

Let $G = (V, E)$ be an undirected graph, with set of vertices V, and set of required edges E_R. We call non-required edges to those in $E_{NR} = E \setminus E_R$. Each $e \in E_R$ has a service cost $c_e^s \geq 0$ and each non-required edge $e \in E_{NR}$ has a deadheading cost $c_e \geq 0$, and $c_e^s \geq c_{e'}$ is assumed. To model the fact that a required edge can be traversed while being serviced (with cost c_e^s) and without being served (with cost c_e), we assume that each required edge has a non-required one, denoted as $e' \in E'_{NR}$, in parallel. We will call $E''_{NR} = E_{NR} \setminus E'_{NR}$. The goal of the LC K-RPP is to find K tours (closed walks starting and ending at a special vertex called the depot) with length no greater than a given value L that jointly traverse (and service) all the required edges, with minimum total cost.

The LC K-RPP is formulated in [3] with a binary variable x_e^k for each edge $e \in E_R$ and for each vehicle $k \in \{1, \ldots, K\}$, and two binary variables x_e^k and y_e^k for each edge $e \in E_{NR}$ and for each vehicle $k \in \{1, \ldots, K\}$. Variable x_e^k for each edge $e \in E_R$ takes the value 1 if e is traversed (and serviced) by vehicle k and 0 otherwise. Variables x_e^k and y_e^k for each edge $e \in E_{NR}$ take the value 1 if e is traversed once or twice, respectively, by vehicle k and 0 otherwise. In other words, variables x_e^k and y_e^k represent the first and second traversal, respectively, of the non-required edge e by vehicle k. The use of these variables is inspired by the work in [7] for the *Maximum Benefit Chinese Postman Problem* (MBCPP).

There are 3 variables for each vehicle between two endpoints i, j of a required edge e because it is necessary to distinguish among traversing e while serving it (with a cost c_e^s) and deadheading once (or twice) from i to j (with a cost $c_{e'} \leq c_e^s$). Although in all the optimal solutions the three variables will never be non-zero simultaneously (see [3]), the three variables are needed to formulate the problem.

As with other routing problems with several vehicles, determining the dimension of the polyhedron defined as the convex hull of the LC K-RPP solutions is a very difficult task, because the constraints that limit the length of each route. However, if they are removed, the problem results the *K vehicles Rural Postman Problem* (K-RPP), whose polyhedron is studied in [3]. In the proofs of the study of this polyhedron for the general case $K \geq 2$ some polyhedral results from the 1-RPP are needed. This is the purpose of this paper.

More specifically, in Section 2, the new formulation for 1-RPP is presented and the polytope associated with its feasible solutions is defined. In addition, the dimension of this polytope is obtained and the facet-inducing property of some inequalities in the formulation is proved. The study of parity inequalities is presented in Section 3, while sections 4 and 5 are devoted to p-connectivity and K-C inequalities, respectively. Finally, some conclusions are presented in Section 6.

2 The 1-RPP polyhedron

The K-RPP for $K = 1$, or 1-RPP, is the Rural Postman Problem with some special features. First, it is defined on a graph that has a non-required edge parallel to each required one and, second, the problem is formulated with three variables associated with the traversal of a required edge e and its parallel non-required one e'.

Different formulations for the RPP have been proposed and their associated polyhedra have been studied in [9], [10], [12], [6], and [19]. In what follows we present the K-RPP formulation in [3] adapted to the case $K = 1$.

Consider an undirected and connected graph $G = (V, E)$, with $E = E_R \cup E'_{NR} \cup E''_{NR}$, where E'_{NR} is the set of non-required edges parallel to an edge in E_R, and where the set V_R formed with the vertices incident with some edge in E_R plus the depot (vertex 1), is not necessarily equal to V. The following notation is used. Given two subsets of vertices $S, S' \subseteq V$, $(S : S')$ denotes the edge set with one endpoint in S and the other one in S'. Given a subset $S \subseteq V$, let us denote $\delta(S) = (S : V\setminus S)$ and $E(S) = (S : S)$. For any subset $F \subseteq E$, we will denote $F_R = F \cap E_R$ and $F_{NR} = F \cap E_{NR}$.

The 1-RPP is formulated with the following variables: x_e, $\forall e \in E_R$, which takes the value 1 if e is serviced, and variables x_e and y_e, representing the first and second traversal, respectively, of the non-required edge e.

$$\text{Minimize} \qquad \sum_{e\in E_R} c_e^s x_e + \sum_{e\in E_{NR}} c_e\big(x_e + y_e\big)$$

$$\sum_{e\in\delta_R(i)} x_e + \sum_{e\in\delta_{NR}(i)} \big(x_e + y_e\big) \equiv 0 \pmod 2, \qquad \forall i\in V \tag{1}$$

$$\sum_{e\in\delta_R(S)} x_e + \sum_{e\in\delta_{NR}(S)} \big(x_e + y_e\big) \geq 2x_f, \qquad \forall S\subseteq V\setminus\{1\}, \forall f\in E(S), \tag{2}$$

$$x_e = 1, \qquad \forall e \in E_R \tag{3}$$

$$x_e \geq y_e, \qquad \forall e\in E_{NR} \tag{4}$$

$$x_e, y_e \in \{0, 1\}, \qquad \forall e\in E_{NR}. \tag{5}$$

Constraints (1) force the route to visit each vertex an even number of times, possibly zero. Conditions (2) ensure the the route is connected and connected to the depot (represented as vertex 1). The traversal of all the required edges is ensured by constraints (3). Constraints (4) guarantee that a second traversal of a non required edge can only occur when it has been traversed previously. Constraints (5) are the binary conditions for the variables.

Note that $x_e = 1$ $\forall e \in E_R$ in any 1-RPP tour and $y_{e'} = 0$ $\forall e' \in E'_{NR}$ in all the optimal ones. Hence, these variables could be removed from the formulation. However, since this is not true for $K > 1$, we will keep these variables because they are necessary in the proofs of the polyhedral study of the K-RPP(G) for $K > 1$. Hence, we will accept feasible (but not optimal) solutions with some variables $y_{e'} = 1$.

Let us call 1-RPP tour to a closed walk on graph G starting and ending at the depot and servicing all the required edges. Associated with each 1-RPP tour we can consider:

(a) An incidence vector $(x, y) \in \mathbb{Z}^{2|E_{NR}|+|E_R|}$, where variables x_e take the value 1 if edge e is traversed once, variables y_e take the value 1 if edge e is traversed twice, and

(b) a support graph $(V, E^{(x,y)})$, where $E^{(x,y)}$ contains one copy of edge $e \in E$ for each variable $x_e = 1$ or $y_e = 1$.

Note that the support graphs are even and connected. Conversely, any even and connected subgraph of G corresponds to a tour on G. In fact, an incidence vector or a subgraph may correspond to several different closed walks, but all of them have the same cost and they can be easily computed (with the Hierholzer algorithm [14], for example). Hence, and for the sake of simplicity, we will call 1-RPP tour on G either to the closed walk, to its incidence vector, and to its corresponding support graph.

The polyhedron **1-RPP**(G) is defined as the convex hull of all the 1-RPP tours in G. To its study, we need some results presented in [7] for the MBCPP. In [7], the MBCPP is defined in a general setting (considering several benefits for each edge), and transformed later into the following simplified version. Given an undirected connected graph $G = (V, E)$, where $1 \in V$ represents the depot, with two benefits for each edge $e \in E$ associated with the first and the second traversals of e, respectively, the MBCPP consists of finding a tour starting from the depot, traversing some of the edges in E at most twice and returning to the depot, with maximum total benefit. The MBCPP is formulated with two binary variables x_e and y_e for each edge $e \in E$ representing the first and second traversal of e, respectively. It is shown that the convex hull of all the MBCPP tours, i.e., the vectors (x, y) satisfying

$$\sum_{e\in\delta(i)} (x_e + y_e) \equiv 0 \pmod 2, \qquad \forall i \in V \tag{6}$$

$$\sum_{e\in\delta(S)} (x_e + y_e) \geq 2x_f, \qquad \forall S \subset V \setminus \{1\}, \;\; \forall f \in E(S) \tag{7}$$

$$x_e \geq y_e, \qquad \forall e \in E \tag{8}$$

$$x_e, y_e \in \{0, 1\}, \qquad \forall e \in E, \tag{9}$$

is a full dimensional polytope and several families of valid and facet-inducing inequalities are described.

In this paper we study the 1-RPP formulated with only one variable associated with each required edge, while, if we consider the MBCPP on the same graph, we have two variables for each edge, including the required ones. Nevertheless, each 1-RPP solution $(x^1, y^1) \in \mathbb{Z}^{(2|E_{NR}|+|E_R|)}$, is a closed walk starting and ending at the depot, and it can be completed with variables $y_e = 0$ for each $e \in E_R$ to obtain a MBCPP tour. Hence we have the following theorem:

Theorem 1 *Let $f(x, y) \geq \alpha$ a valid inequality for the MBCPP on graph G. By removing all the variables y_e, $e \in E_R$, the resulting inequality $f(x, y) \geq \alpha$ is valid for the 1-RPP.*

For example, from inequalities (7) we obtain inequalities (2). Furthermore, from several families of valid inequalities for the MBCPP, namely parity, p-connectivity and K-C inequalities, we will obtain valid inequalities for the 1-RPP (see Sections 3, 4, and 5).

In the following, we will obtain the dimension of 1-RPP(G) and will study conditions under which some of the above constraints and other valid inequalities define facets of it. For this study, we need to build several 1-RPP tours in graph G. For example, the graph formed with two copies of each edge in E_{NR} and then replacing one copy of each $e \in E'_{NR}$ by the required edge parallel to e, is a 1-RPP tour. This basic tour is used in the proof of Theorem 2. In the proofs of other theorems, we will build more specific and detailed 1-RPP tours. In order to do this, we need some more definitions.

Consider the (generally disconnected) subgraph (V_R, E_R) of G. We call a connected component of this subgraph an R-connected component. Note that a R-connected component may consist only of the depot. A vertex not belonging to any R-connected component (a vertex which is not incident with any required edge) will be called an isolated vertex.

Given a vertex subset $V_o \subseteq V$, with $|V_o|$ even, a subset of edges $M \subseteq E$ is a T-join if, in the subgraph (V, M), the degree of v is odd if, and only if, $v \in V_o$. It is known that, if G is connected, it has a T-join for each set $V_o \subseteq V$, with $|V_o|$ even (see [17], for instance).

Given $G = (V, E) = (V, E_R \cup E'_{NR} \cup E''_{NR})$, let $V_o^R \subseteq V$ be the set of R-odd vertices, i.e., vertices incident with an odd number of required edges. Let $M \subseteq E_{NR}$ be any corresponding T-join. The set of edges $M \cup E_R$ form an even graph, although not necessarily connected. If we add the edges in a closed walk starting at the depot, visiting at least one node in each connected component of $M \cup E_R$ and ending at the depot, we obtain a 1-RPP tour.

Theorem 2 *If (V, E_{NR}) is a 3-connected graph, then dim(1-RPP(G))$= 2|E_{NR}|$.*

Proof: 1-RPP(G) is a polytope in $\mathbb{R}^{2|E_{NR}|+|E_R|}$. Since all its points satisfy equations (3), which are linearly independent, we have dim(1-RPP(G))$\leq 2|E_{NR}|$. We will prove that dim(1-RPP(G))$\geq 2|E_{NR}|$. Let $ax + by = c$, i.e.,

$$\sum_{e\in E_R} a_e x_e + \sum_{e\in E_{NR}} a_e x_e + \sum_{e\in E_{NR}} b_e y_e = c \tag{10}$$

be an equation satisfied by all the 1-RPP tours. We have to prove that this equation is a linear combination of equations (3), i.e., to prove that

$$\begin{aligned} a_e &= 0, \quad \forall e \in E_{NR}, \\ b_e &= 0, \quad \forall e \in E_{NR}, \\ c &= \sum_{e\in E_R} a_e. \end{aligned}$$

Let $T = (x, y)$ be the 1-RPP tour formed with two copies of each edge in E_{NR} and then replacing one copy of each $e \in E'_{NR}$ by the required edge parallel to e. T has all its entries equal to 1 except for $y_e = 0, \forall e \in E'_{NR}$ and, by substituting it in (10), we obtain

$$\sum_{e\in E_R} a_e + \sum_{e\in E_{NR}} a_e + \sum_{e\in E''_{NR}} b_e = c. \tag{11}$$

Let be $f \in E''_{NR}$. Since (V, E_{NR}) is a 3-connected graph, the tour T^{-2f} obtained after removing from T the two copies of f is also a 1-RPP tour. Hence, by substituting it in (10), we obtain

$$\sum_{e\in E_R} a_e + \sum_{e\in E_{NR}\setminus\{f\}} a_e + \sum_{e\in E''_{NR}\setminus\{f\}} b_e = c,$$

and by subtracting this equation from (11), we obtain $a_f + b_f = 0$ for all $f \in E''_{NR}$.

Let $\mathcal{C}$ be an arbitrary cycle on G formed by non-required edges. Let us recall that $\mathcal{C}' = \mathcal{C} \cap E'_{NR}$ and $\mathcal{C}'' = \mathcal{C} \cap E''_{NR}$. If we remove from T one copy of each edge in $\mathcal{C}$, we obtain another 1-RPP tour $T - \mathcal{C}$. After substituting it in (10) and subtracting the corresponding equation from (11), we obtain $a(\mathcal{C}') + b(\mathcal{C}'') = 0$. On the other hand, if we add to T one copy of each edge in $\mathcal{C}$ and, then, we remove two copies of each edge appearing three times, we obtain another 1-RPP tour $T + \mathcal{C}$. After substituting it in (10) and subtracting the corresponding equation from (11), we obtain $b(\mathcal{C}') - b(\mathcal{C}'') = 0$, and, as $a_f + b_f = 0$ for all $f \in E''_{NR}$, $b(\mathcal{C}') + a(\mathcal{C}'') = 0$. Hence, for each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ on G formed by non-required edges,

$$a(\mathcal{C}') + b(\mathcal{C}'') = 0. \tag{12}$$

$$b(\mathcal{C}') + a(\mathcal{C}'') = 0. \tag{13}$$

Let $f = (i,j) \in E''_{NR}$. Since (V, E_{NR}) is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from $f = (i,j)$. Consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$, for which equation (12) holds. Hence, we have:

$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + b_f = 0$, $a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + b_f = 0$, and $a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$, and we obtain $b_f = 0$. As $a_f + b_f = 0$ holds, we have $a_f = b_f = 0$ for all $f \in E''_{NR}$.

Let $f = (i,j) \in E'_{NR}$. Let $\mathcal{P}_1$, $\mathcal{P}_2$ two edge-disjoint paths as above and consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. From equation (12) we have:

$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a_f = 0$, $a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + a_f = 0$, and $a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$, from which we obtain $a_f = 0$, and from (13) we also have:

$b(\mathcal{P}'_1) + a(\mathcal{P}''_1) + b_f = 0$, $b(\mathcal{P}'_2) + a(\mathcal{P}''_2) + b_f = 0$, and $b(\mathcal{P}'_1) + a(\mathcal{P}''_1) + b(\mathcal{P}'_2) + a(\mathcal{P}''_2) = 0$, from which we obtain $b_f = 0$. Hence, we have $a_f = b_f = 0$ for all $f \in E'_{NR}$.

Hence, $a_e = b_e = 0$ for all $e \in E_{NR}$ and, by substituting in (11) we obtain $\sum_{e \in E_R} a_e = c$ and we are done. ♦

The result in Theorem 2 is not true when graph (V, E_{NR}) is not 3-connected. For example, if $|V| \leq 3$ and E_{NR} defines a complete graph, it can be seen that dim(1-RPP(G))= $2|E_{NR}| - 2$.

In what follows, in order to prove that some inequalities are facet-defining of 1-RPP(G), we will assume that (V, E_{NR}) is a 3-connected graph.

Theorem 3 *Inequality $y_e \geq 0$, for each edge $e \in E_{NR}$, is facet-inducing of 1-RPP(G).*

Proof: Let $ax + by \geq c$, i.e., $\sum_{f \in E_R} a_f x_f + \sum_{f \in E_{NR}} a_f x_f + \sum_{f \in E_{NR}} b_f y_f \geq c$, be a valid inequality such that $\{(x,y) \in \text{1-RPP}(G) : \; y_e = 0\} \subseteq \{(x,y) \in \text{1-RPP}(G) : \; ax + by = c\}$. We have to prove that this inequality is a linear combination of the equalities (3) and $y_e \geq 0$. Note that this means to prove that

$$\begin{aligned} a_f &= 0, \quad \forall f \in E_{NR}, \\ b_f &= 0, \quad \forall f \in E_{NR}, \; f \neq e, \\ c &= \sum_{f \in E_R} a_f. \end{aligned}$$

(a) We will first prove it for $e \in E''_{NR}$. Consider the tour T^{-2e} of the proof of Theorem 2, which satisfies $y_e = 0$ and, hence, $ax + by = c$ holds. By replacing the incidence vector in

$ax + by = c$ we obtain

$$\sum_{f \in E_R} a_f + \sum_{f \in E_{NR} \setminus \{e\}} a_f + \sum_{f \in E''_{NR} \setminus \{e\}} b_f = c. \tag{14}$$

Let be $g \in E''_{NR}$, $g \neq e$. Since (V, E_{NR}) is a 3-connected graph, the tour T^{-2e-2g} obtained after removing from T^{-2e} the two copies of g is also a 1-RPP tour satisfying $y_e = 0$. Hence, by substituting it in $ax + by = c$ and by subtracting the corresponding equality from (14) we obtain that $a_g + b_g = 0,\ \forall g \in E''_{NR},\ g \neq e$.

Let be $g \in E'_{NR}$. Since (V, E_{NR}) is a 3-connected graph, $(V, E_{NR} \setminus \{e, g\})$ is a connected graph, and there is a T-join in $(V, E_{NR} \setminus \{e, g\})$ connecting the R-odd vertices. The 1-RPP tour T^* build wit this T-join does not traverse g and satisfies $y_e = 0$. Furthermore, the tour obtained after adding to T^* the two copies of g is also a 1-RPP tour satisfying $y_e = 0$. Hence, by substituting them in $ax + by = c$ and by subtracting the corresponding equalities we obtain that $a_g + b_g = 0,\ \forall g \in E'_{NR}$.

For each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ on G formed by non-required edges, the tour $T^{-2e} + \mathcal{C}$ obtained after adding to T^{-2e} one copy of each edge in $\mathcal{C}$ (and, then, removing two copies of any edge traversed three times) is also a 1-RPP tour satisfying $y_e = 0$. Proceeding as in the proof of Theorem 2 we obtain $b(\mathcal{C}') - b(\mathcal{C}'') = 0$ and, given that $a_g + b_g = 0,\ \forall g \in E''_{NR},\ g \neq e$, we obtain that $b(\mathcal{C}') + a(\mathcal{C}'' \setminus \{e\}) - b_e = 0$ if $e \in \mathcal{C}$ and $b(\mathcal{C}') + a(\mathcal{C}'') = 0$ if $e \notin \mathcal{C}$.

Let be $f = (i, j) \in E''_{NR}$. Since (V, E_{NR}) is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from f. Let us assume, for instance, that $e \in \mathcal{P}_1$. Consider the two cycles $\mathcal{P}_1 \cup \{f\}$ and $\mathcal{P}_1 \cup \mathcal{P}_2$, for which equation $b(\mathcal{C}') + a(\mathcal{C}'' \setminus \{e\}) - b_e = 0$ holds, and the cycle $\mathcal{P}_2 \cup \{f\}$, for which equation $b(\mathcal{C}') + a(\mathcal{C}'') = 0$ holds. Hence, we have:

$b(\mathcal{P}'_1) + a(\mathcal{P}''_1 \setminus \{e\}) - b_e + a_f = 0,$

$b(\mathcal{P}'_1) + a(\mathcal{P}''_1 \setminus \{e\}) - b_e + b(\mathcal{P}'_2) + a(\mathcal{P}''_2) = 0,$ and

$b(\mathcal{P}'_2) + a(\mathcal{P}''_2) + a_f = 0,$

and we obtain $a_f = 0$. Furthermore, if $f \neq e$, $a_f + b_f = 0$ holds, we have $b_f = 0$ for all $f \in E''_{NR} \setminus \{e\}$. The case $e \notin \mathcal{P}_1 \cup \mathcal{P}_2$ is similar.

Let be $f = (i, j) \in E'_{NR}$. Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two edge-disjoint paths as above and assume in this case that $e \notin \mathcal{P}_1$ and $e \notin \mathcal{P}_2$. Consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. Now we have:

$b(\mathcal{P}'_1) + a(\mathcal{P}''_1) + b_f = 0,\ \ b(\mathcal{P}'_2) + a(\mathcal{P}''_2) + b_f = 0,$ and $\ b(\mathcal{P}'_1) + a(\mathcal{P}''_1) + b(\mathcal{P}'_2) + a(\mathcal{P}''_2) = 0,$ from which we obtain $b_f = 0$. Furthermore, as $a_f + b_f = 0,\ \forall f \in E'_{NR}$ we obtain that $a_f = 0$.

Hence, $a_f = b_f = 0$ for all $f \in E_{NR} \setminus \{e\}$ and $a_e = 0$ and, by replacing them in (14) we obtain $\sum_{f \in E_R} a_f = c$ and we are done.

(b) We will prove it now for $e \in E'_{NR}$. The tour T of the proof of Theorem 2 satisfies $y_e = 0$ and, hence, $ax + by = c$ holds, and by replacing the incidence vector we obtain equation (11):

$$\sum_{e \in E_R} a_e + \sum_{e \in E_{NR}} a_e + \sum_{e \in E''_{NR}} b_e = c.$$

Let be $f \in E''_{NR}$. Since (V, E_{NR}) is a 3-connected graph, the tour T^{-2f} is also a 1-RPP tour satisfying $y_e = 0$. Hence, by substituting it in $ax + by = c$ and by subtracting the corresponding equality from (11) we obtain that $a_f + b_f = 0$, $\forall f \in E''_{NR}$.

Let be $f \in E'_{NR}$, $f \neq e$. Since (V, E_{NR}) is a 3-connected graph, $(V, E_{NR} \setminus \{e, f\})$ is a connected graph, and there is a T-join in $(V, E_{NR} \setminus \{e, f\})$ connecting the R-odd vertices. The 1-RPP tour T^* build wit this T-join does nos traverses f and satisfies $y_e = 0$. Furthermore, the tour obtained after adding to T^* the two copies of f is also a 1-RPP tour satisfying $y_e = 0$. Hence, by subtracting the corresponding equalities we obtain that $a_f + b_f = 0$, $\forall f \in E'_{NR}$, $f \neq e$.

For each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ on G formed by non-required edges, the tour $T - \mathcal{C}$ obtained after subtracting from T one copy of each edge in $\mathcal{C}$ is also a 1-RPP tour satisfying $y_e = 0$. Proceeding as in the proof of Theorem 2 we obtain $a(\mathcal{C}') + b(\mathcal{C}'') = 0$.

Let be $f = (i, j) \in E''_{NR}$. Since (V, E_{NR}) is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from f. Considering the three cycles as in (a) we obtain $b_f = 0$ and, as $a_f + b_f = 0$ holds, we have $a_f = b_f = 0$ for all $f \in E''_{NR}$.

Let be $f = (i, j) \in E'_{NR}$. Proceeding as above we obtain $a_f = 0$ and, hence, also $b_f = 0$ $\forall f \in E'_{NR} \setminus \{e\}$. We have $a_f = b_f = 0$ for all $f \in E_{NR} \setminus \{e\}$ and $a_e = 0$ and, by replacing them in (14) we obtain $\sum_{f \in E_R} a_f = c$ and we are done. ♦

Theorem 4 *Inequality $x_e \leq 1$, for each edge $e \in E_{NR}$, is facet-inducing for 1-RPP(G).*

Proof: Let $ax + by \leq c$, i.e., $\displaystyle\sum_{f \in E_R} a_f x_f + \sum_{f \in E_{NR}} a_f x_f + \sum_{f \in E_{NR}} b_f y_f \leq c$, be a valid inequality such that $\{(x, y) \in \text{1-RPP}(G) : \quad x_e = 1\} \subseteq \{(x, y) \in \text{1-RPP}(G) : \quad ax + by = c\}$. We have to prove that this inequality is a linear combination of the equalities (3) and $x_e \leq 1$. Note that this means to prove that

$$\begin{aligned} a_f &= 0, \quad \forall f \in E_{NR},\ f \neq e, \\ b_f &= 0, \quad \forall f \in E_{NR}, \\ c &= \sum_{f \in E_R} a_f + a_e. \end{aligned}$$

(a) We will first prove it for $e \in E''_{NR}$. Consider the tour T of the proof of Theorem 2, which satisfies $x_e = 1$ and, hence, $ax + by = c$ holds. By replacing the incidence vector in $ax + by = c$ we obtain

$$\sum_{f \in E_R} a_f + \sum_{f \in E_{NR}} a_f + \sum_{f \in E''_{NR}} b_f = c. \tag{15}$$

Let be $f \in E''_{NR}$, $f \neq e$. Since (V, E_{NR}) is a 3-connected graph, the tour T^{-2f} is also a 1-RPP tour satisfying $x_e = 1$. Hence, by substituting it in $ax + by \leq c$ and by subtracting the corresponding equality from (15) we obtain that $a_f + b_f = 0$, $\forall f \in E''_{NR}$, $f \neq e$.

Let be $f \in E'_{NR}$. Since (V, E_{NR}) is a 3-connected graph, $(V, E_{NR} \setminus \{e, f\})$ is a connected graph, and there is a T-join in $(V, E_{NR} \setminus \{e, f\})$ connecting the R-odd vertices. The 1-RPP tour T^* build wit this T-join does not traverse f nor e. T^{*+2e} is also a 1-RPP tour and satisfies $x_e = 1$. Furthermore, $T^{*+2e+2f}$ is also a 1-RPP tour satisfying $x_e = 1$. Hence, by substituting them in $ax + by = c$ and by subtracting the corresponding equalities we obtain that $a_f + b_f = 0,\ \forall f \in E'_{NR}$.

For each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ on G formed by non-required edges, the tour $T - \mathcal{C}$ obtained after removing from T one copy of each edge in $\mathcal{C}$ is also a 1-RPP tour satisfying $x_e = 1$. Proceeding as in the proof of Theorem 2 we obtain $a(\mathcal{C}') + b(\mathcal{C}'') = 0$.

Let be $f = (i, j) \in E''_{NR}$. Since (V, E_{NR}) is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from f. Consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$, for which equation $a(\mathcal{C}') + b(\mathcal{C}'') = 0$ holds. Hence, we have:

$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + b_f = 0,\ \ a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + b_f = 0$, and $\ a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$,
and we obtain $b_f = 0$. Furthermore, if $f \neq e$, $a_f + b_f = 0$ holds, we have $a_f = 0$ for all $f \in E''_{NR} \setminus \{e\}$.

Let be $f = (i, j) \in E'_{NR}$. Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two edge-disjoint paths as above and consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. Now we have:

$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a_f = 0,\ \ a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + a_f = 0$, and $\ a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$,
from which we obtain $a_f = 0$. Furthermore, as $a_f + b_f = 0,\ \forall f \in E'_{NR}$ we obtain that $b_f = 0$.

Hence, $a_f = b_f = 0$ for all $f \in E_{NR} \setminus \{e\}$ and $b_e = 0$ and, by replacing them in (15) we obtain $\sum_{f \in E_R} a_f + a_e = c$ and we are done.

(b) We will prove it now for $e \in E'_{NR}$. Consider the tour T of the proof of Theorem 2, which satisfies $x_e = 1$ and, hence, $ax + by = c$ holds. By replacing the incidence vector in $ax + by = c$ we obtain

$$\sum_{f \in E_R} a_f + \sum_{f \in E_{NR}} a_f + \sum_{f \in E''_{NR}} b_f = c. \tag{16}$$

Let be $f \in E''_{NR}$. Since (V, E_{NR}) is a 3-connected graph, the tour T^{-2f} obtained after removing from T the two copies of f is also a 1-RPP tour satisfying $x_e = 1$. Hence, by substituting it in $ax + by = c$ and by subtracting the corresponding equality from (16) we obtain that $a_f + b_f = 0,\ \forall f \in E''_{NR}$.

Let be $f \in E'_{NR}$, $f \neq e$. Since (V, E_{NR}) is a 3-connected graph, $(V, E_{NR} \setminus \{e, f\})$ is a connected graph, and there is a T-join in $(V, E_{NR} \setminus \{e, f\})$ connecting the R-odd vertices. The 1-RPP tour T^* build wit this T-join does not traverse f nor e. T^{*+2e} is also a 1-RPP tour and satisfies $x_e = 1$. Furthermore, $T^{*+2e+2f}$ is also a 1-RPP tour satisfying $x_e = 1$. Hence, by substituting them in $ax + by = c$ and by subtracting the corresponding equalities we obtain that $a_f + b_f = 0,\ \forall f \in E'_{NR}$, $f \neq e$.

For each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ on G formed by non-required edges, the tour $T + \mathcal{C}$ obtained after adding to T one copy of each edge in $\mathcal{C}$ (and, then, removing two copies of any edge traversed three times) is also a 1-RPP tour satisfying $x_e = 1$. Proceeding as in the proof of Theorem 2 we obtain $b(\mathcal{C}') - b(\mathcal{C}'') = 0$ and, given that $a_f + b_f = 0,\ \forall f \in E''_{NR}$, we obtain that $b(\mathcal{C}') + a(\mathcal{C}'') = 0$.

Let be $f = (i, j) \in E''_{NR}$. Since (V, E_{NR}) is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from f. Consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$, for which equation $b(\mathcal{C}') + a(\mathcal{C}'') = 0$ holds. Hence, we have:

$b(\mathcal{P}_1') + a(\mathcal{P}_1'') + a_f = 0$, $b(\mathcal{P}_2') + a(\mathcal{P}_2'') + a_f = 0$, and $b(\mathcal{P}_1') + a(\mathcal{P}_1'') + b(\mathcal{P}_2') + a(\mathcal{P}_2'') = 0$, and we obtain $a_f = 0$. Furthermore, as $a_f + b_f = 0$ holds, we have $a_f = 0$ for all $f \in E''_{NR}$.

Let be $f = (i,j) \in E'_{NR}$. Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two edge-disjoint paths as above and consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. Now we have:

$b(\mathcal{P}_1') + a(\mathcal{P}_1'') + b_f = 0$, $b(\mathcal{P}_2') + a(\mathcal{P}_2'') + b_f = 0$, and $b(\mathcal{P}_1') + a(\mathcal{P}_1'') + b(\mathcal{P}_2') + a(\mathcal{P}_2'') = 0$, from which we obtain $b_f = 0$. Furthermore, if $f \neq e$ as $a_f + b_f = 0$, $\forall f \in E'_{NR} \setminus \{e\}$ we obtain that $a_f = 0$.

Hence, $a_f = b_f = 0$ for all $f \in E_{NR} \setminus \{e\}$ and $b_e = 0$ and, by replacing them in (16) we obtain $\sum_{f \in E_R} a_f + a_e = c$ and we are done. ♦

Theorem 5 *Inequalities (4), $x_e \geq y_e$ for every edge $e \in E_{NR}$, are facet-inducing for 1-RPP(G) if graph $(V, E_{NR} \setminus \{e\})$ is 3-edge connected.*

Proof: Let $ax + by \geq c$, i.e., $\displaystyle\sum_{f \in E_R} a_f x_f + \sum_{f \in E_{NR}} a_f x_f + \sum_{f \in E_{NR}} b_f y_f \geq c$, be a valid inequality such that $\{(x,y) \in \text{1-RPP}(G): \; x_e = y_e\} \subseteq \{(x,y) \in \text{1-RPP}(G): \; ax + by = c\}$. We have to prove that this inequality is a linear combination of the equalities (3) and $x_e - y_e \geq 0$. Note that this means to prove that

$$
\begin{aligned}
a_f = b_f &= 0, \;\; \forall f \in E_{NR}, \; f \neq e, \\
a_e &= -b_e, \\
c &= \sum_{f \in E_R} a_f.
\end{aligned}
$$

(a) We will first prove it for $e \in E''_{NR}$. Consider the tour T of the proof of Theorem 2, which satisfies $x_e = y_e (= 1)$ and, hence, $ax + by = c$ holds. By replacing the incidence vector in $ax + by = c$ we obtain

$$
\sum_{f \in E_R} a_f + \sum_{f \in E_{NR}} a_f + \sum_{f \in E''_{NR}} b_f = c. \tag{17}
$$

Let be $f \in E''_{NR}$ (including $f = e$). Since (V, E_{NR}) is a 3-connected graph, vector T^{-2f} is also a 1-RPP tour and it satisfies $x_e = y_e$. Hence, by replacing it in $ax + by = c$ and then subtracting the corresponding equation from (17) we obtain that $a_f + b_f = 0$ for all $f = (i,j) \in E''_{NR}$.

Let be $f \in E'_{NR}$. Since (V, E_{NR}) is a 3-connected graph, $(V, E_{NR} \setminus \{e, f\})$ is a connected graph, and there is a T-join in $(V, E_{NR} \setminus \{e, f\})$ connecting the R-odd vertices. The 1-RPP tour T^* build wit this T-join does not traverse f nor e and then satisfies $x_e = y_e$. T^{*+2f} is also a 1-RPP satisfying $x_e = y_e$. By substituting them in $ax + by = c$ and by subtracting the corresponding equalities we obtain that $a_f + b_f = 0$, $\forall f \in E'_{NR}$.

For each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ on $G \setminus \{e\}$ formed by non-required edges, the tour $T - \mathcal{C}$ obtained after removing from T one copy of each edge in $\mathcal{C}$ is also a 1-RPP tour satisfying $x_e = y_e$. Proceeding as in the proof of Theorem 2 we obtain $a(\mathcal{C}') + b(\mathcal{C}'') = 0$.

Let be $f = (i,j) \in E''_{NR}$, $f \neq e$. Since $(V, E_{NR} \setminus \{e\})$ is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from e and also different from f. Consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$, for which equation $a(\mathcal{C}') + b(\mathcal{C}'') = 0$ holds. Hence, we have:

$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + b_f = 0$, $a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + b_f = 0$, and $a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$, and we obtain $b_f = 0$. Furthermore, as $a_f + b_f = 0$ holds, we have $a_f = 0$ for all $f \in E''_{NR} \setminus \{e\}$.

Let be $f = (i,j) \in E'_{NR}$. Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two edge-disjoint paths as above and consider the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. Now we have:

$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a_f = 0$, $a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + a_f = 0$, and $a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$, from which we obtain $a_f = 0$ and, as $a_f + b_f = 0$ holds, we have $b_f = 0$ for all $f \in E'_{NR}$.

Hence, $a_f = b_f = 0$ for all $f \in E_{NR} \setminus \{e\}$ and $a_e + b_e = 0$ and, by replacing them in (17) we obtain $\sum_{f \in E_R} a_f = c$ and we are done.

(b) We will prove it now for $e \in E'_{NR}$. Since (V, E_{NR}) is a 3-connected graph, $(V, E_{NR} \setminus \{e\})$ is a connected graph, and there is a T-join in $(V, E_{NR} \setminus \{e\})$ connecting the R-odd vertices. The 1-RPP tour T^* build wit this T-join does not traverse e and then satisfies $x_e = y_e (= 0)$. Furthermore, T^{*+2e} is also a 1-RPP satisfying $x_e = y_e (= 1)$. By substituting them in $ax + by = c$ and by subtracting the corresponding equalities we obtain that $a_e + b_e = 0$.

Let be $f \in E'_{NR} \cup E''_{NR}$, $f \neq e$. Since (V, E_{NR}) is a 3-connected graph, $(V, E_{NR} \setminus \{e, f\})$ is a connected graph, and there is a T-join in $(V, E_{NR} \setminus \{e, f\})$ connecting the R-odd vertices. The 1-RPP tour T^{**} build with this T-join does not traverse f nor e and then satisfies $x_e = y_e (= 0)$. Furthermore, T^{**+2f} is also a 1-RPP satisfying $x_e = y_e = 0$. By substituting them in $ax + by = c$ and by subtracting the corresponding equalities we obtain that $a_f + b_f = 0$. Hence, we have $a_f + b_f = 0 \ \forall f \in E_{NR}$.

For each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ on $G \setminus \{e\}$ formed by non-required edges, the tour $T^* + \mathcal{C}$ obtained after adding to T^* one copy of each edge in $\mathcal{C}$ (and, then, removing two copies of any edge traversed three times) is also a 1-RPP tour satisfying $x_e = y_e$. Proceeding as in the proof of Theorem 2 we obtain $b(\mathcal{C}') - b(\mathcal{C}'') = 0$ and, given that $a_f + b_f = 0$, $\forall f \in E_{NR}$, $b(\mathcal{C}') + a(\mathcal{C}'') = 0$ holds.

For each $f = (i,j) \in E_{NR}$, $f \neq e$, since $(V, E_{NR} \setminus \{e\})$ is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from e and f. By considering the three cycles $\mathcal{P}_1 \cup \{f\}$, $\mathcal{P}_2 \cup \{f\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$, for which equation $b(\mathcal{C}') + a(\mathcal{C}'') = 0$ holds, we obtain $b_f = 0$ if $f \in E'_{NR}$ and $a_f = 0$ if $f \in E''_{NR}$, and, hence, $a_f = b_f = 0$ for all $f \in E_{NR} \setminus \{e\}$. By replacing this and $a_e + b_e = 0$ in (17) we obtain $\sum_{f \in E_R} a_f = c$ and we are done. ♦

Note 1 Inequalities $x_e \geq 0$ and $y_e \leq 1$, for all $e \in E_{NR}$, are not facet-inducing because they are dominated by inequalities $x_e \geq y_e$.

In what follows we describe conditions under which connectivity inequalities (2) are facet inducing. Note that, given that $x_e = 1$ for all $e \in E_R$, if $\delta_R(S) \neq \emptyset$ then inequalities (2) are obviously satisfied. Hence, we will assume $\delta_R(S) = \emptyset$. Furthermore, if $E(S)$ contains some required edge f, as $x_f = 1$ holds, inequalities (2) are dominated by inequalities

$$\sum_{e \in \delta(S)} (x_e + y_e) \geq 2, \quad \forall S \subseteq V \setminus \{1\} : \ \delta_R(S) = \emptyset, \ E_R(S) \neq \emptyset,$$

which are studied in Theorem 7.

Theorem 6 *Let $S \subseteq V \setminus \{1\}$ such that $E_R(S) = \delta_R(S) = \emptyset$. Let $f \in E(S)$ ($f \in E''_{NR}$). The connectivity inequality (2), which now takes the form*

$$(x + y)(\delta(S)) \geq 2x_f, \tag{18}$$

is facet-inducing for 1-RPP(G) if subgraph $(S, E_{NR}(S))$ is 3-edge connected and either $V \backslash S = \{1\}$, or subgraph $(V \backslash S, E_{NR}(V \backslash S))$ is 3-edge connected.

Proof: Let $ax + by \geq c$, i.e., $\sum_{e\in E_R} a_e x_e + \sum_{e\in E_{NR}} a_e x_e + \sum_{e\in E_{NR}} b_e y_e \geq c$, be a valid inequality such that $\{(x,y) \in \text{1-RPP}(G) : \ (x+y)(\delta(S)) - 2x_f = 0\} \subseteq \{(x,y) \in \text{1-RPP}(G) : \ ax+by = c\}$. We have to prove that this inequality is a linear combination of the equalities (3) and $(x + y)(\delta(S)) - 2x_f \geq 0$. Note that this means to prove that

$$\begin{aligned}
a_e = b_e &= 0, \quad \forall e \in E_{NR}(S) \setminus \{f\} \cup E_{NR}(V \setminus S), \\
a_e = b_e &= \alpha, \quad \forall e \in \delta(S), \\
a_f = -2\alpha, & \qquad b_f = 0, \\
c &= \sum_{e\in E_R} a_e.
\end{aligned}$$

Consider now the 1-RPP tour T' formed with two copies of each edge in $E_{NR}(S) \cup E_{NR}(V \setminus S)$ and then replacing one copy of each $e \in E'_{NR}$ by the required edge parallel to e, plus two copies of a given edge $g \in \delta(S)$. Since this tour satisfies $(x + y)(\delta(S)) - 2x_f = 0$, it also satisfies $ax + by = c$, and we have

$$\sum_{e\in E_R} a_e + \sum_{e\in E_{NR}\setminus\delta(S)} a_e + \sum_{e\in E''_{NR}\setminus\delta(S)} b_e + a_g + b_g = c. \tag{19}$$

For each edge $e \in E''_{NR} \backslash \delta(S)$, $e \neq f$, consider the tour T'^{-2e} (it is a 1-RPP tour because subgraphs $G(S)$ and $G(V \setminus S)$ are 2-edge connected). By comparing both tours we obtain $a_e + b_e = 0$ for each edge $e \in E''_{NR} \backslash \delta(S)$, $e \neq f$.

For each edge $e \in E'_{NR} \backslash \delta(S)$, necessarily $e \in E'_{NR}(V \setminus S)$, given that graph $(V \backslash S, E_{NR}(V \setminus S) \setminus \{e\})$ is connected, there is a T-join M connecting its R-odd vertices. The edges in $M \cup E_R$ form an even subgraph in $G(V \setminus S)$, although not necessarily connected. If we add the edges in a closed walk in $E_{NR}(V \setminus S) \setminus \{e\}$ starting at the depot, visiting at least one node in each connected component of $M \cup E_R$ and ending at the depot, we obtain a 1-RPP tour T^* that does not traverse e and satisfies $(x+y)(\delta(S)) = 0 = 2x_f$. The 1-RPP tour T^{*+2e} also satisfies $(x+y)(\delta(S)) = 0$ and, by subtracting the corresponding equations we obtain that $a_e + b_e = 0$, for each edge $e \in E'_{NR} \backslash \delta(S)$.

For each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ either on $G(V \backslash S)$ or in $G(S)$ (traversing edge f or not) formed by non-required edges, the tour $T' - \mathcal{C}$ obtained after removing from T' one copy of each edge in $\mathcal{C}$ is also a 1-RPP tour satisfying $(x + y)(\delta(S)) = 2 = 2x_f$ (note that $x_f = 1$ holds). Proceeding as in the proof of Theorem 2 we obtain $a(\mathcal{C}') + b(\mathcal{C}'') = 0$.

Let $e = (i, j) \in E'_{NR}(V \backslash S)$ (if any). Since that $(V \backslash S, E_{NR}(V \backslash S))$ is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from e. Consider the three cycles $\mathcal{P}_1 \cup \{e\}$, $\mathcal{P}_2 \cup \{e\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$, for which equation $a(\mathcal{C}') + b(\mathcal{C}'') = 0$ holds. Hence, we have:

$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a_e = 0$, $a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + a_e = 0$, and $a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$, and we obtain $a_e = 0$. Furthermore, as $a_e + b_e = 0$ holds, we have $b_e = 0$ for all $e \in E'_{NR}(V \backslash S)$.

Let be $e = (i, j) \in E''_{NR}(V \backslash S)$ (if any). Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two edge-disjoint paths as above and consider the three cycles $\mathcal{P}_1 \cup \{e\}$, $\mathcal{P}_2 \cup \{e\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. Now we have:

$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + b_e = 0$, $a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + b_e = 0$, and $a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$, from which we obtain $b_e = 0$ and, as $a_e + b_e = 0$ holds, we have $a_e = 0$ for all $e \in E''_{NR}(V \backslash S)$.

Hence, $a_e = b_e = 0$ for all $e \in E_{NR}(V \setminus S)$. In a similar way we obtain $a_e = b_e = 0$ for each edge $e \in E_{NR}(S)$, $e \neq f$, and $b_f = 0$.

Let us denote the edges in $\delta(S)$ as $e_1, \ldots, e_p$, where $p \geq 3$ since graph G is 3-edge connected. Now consider two edges $e_1, e_2 \in \delta(S)$. Consider the 1-RPP tour T formed with two copies of each edge in $E_{NR}(S) \cup E_{NR}(V \backslash S)$ and then replacing one copy of each $e \in E'_{NR}$ by the required edge parallel to e, plus two copies of edge e_1. Let T^* be the tour obtained from T after removing the second traversal of e_1 and one copy of each edge of two paths $\mathcal{P}_1, \mathcal{P}_2$ joining the endpoins of e_1 and e_2, and adding the first traversal of e_2. Both tours satisfy $(x + y)(\delta(S)) = 2x_f = 2$ and, after subtracting the corresponding equalities we obtain $a(\mathcal{P}'_1) + b(\mathcal{P}'_2) + a(\mathcal{P}''_1) + b(\mathcal{P}''_2) + b_{e_1} - a_{e_2} = 0$ and, hence, $b_{e_1} = a_{e_2}$. If we interchange the roles of the edges e_1 and e_2, we obtain $b_{e_2} = a_{e_1}$. Proceeding in this way with all the pairs of edges in $\delta(S)$, we obtain $a_{e_i} = b_{e_j}$ for all $i \neq j \in \{1, \ldots, p\}$ and then $a_{e_i} = a_{e_j} = b_{e_i} = b_{e_j}$ for all i, j (because $p \geq 3$ holds).

Let T^* be a 1-RPP tour formed with two copies of each edge in $E_{NR}(V \setminus S)$ and then replacing one copy of each $e \in E'_{NR}$ by the required edge parallel to e, plus two copies of each edge in a given path $\mathcal{P}$ joining a vertex in $G(V \setminus S)$ to an endnode of f traversing $\delta(S)$ once, say, with edge $e \in \delta(S)$, and two copies of edge f. By comparing this tour with the one removing the two copies of the edges in $\mathcal{P}$ and the two copies of f, both satisfying $(x + y)(\delta(S)) = 2x_f$, we obtain that $a_e + b_e + a_f + b_f = 0$. Given that $b_f = 0$ and $a_e = b_e$, we have $a_f = -2a_e$ for any edge $e \in \delta(S)$.

By replacing $a_e = b_e = 0$ for each $e \in E_{NR} \setminus \big(\delta(S) \cup \{f\}\big)$, $a_e = b_e = \alpha$ for each $e \in \delta(S)$, and $b_f = 0$, $a_f = -2\alpha$, in equation (19) we obtain $\sum_{e \in E_R} a_e = c$, and after replacing all the previous facts in $ax + by \geq c$ we obtain

$$\sum_{e \in E_R} a_e x_e + \alpha\Big((x + y)(\delta(S)) - 2x_f\Big) \geq \sum_{e \in E_R} a_e,$$

which is a linear combination of the equalities (3) and $(x + y)(\delta(S)) \geq 2$. Hence, the connectivity inequality (18) is facet-inducing for 1-RPP(G). ♦

Theorem 7 *Let $S \subseteq V \backslash \{1\}$ such that $\delta_R(S) = \emptyset$ and $E_R(S) \neq \emptyset$. The connectivity inequality*

$$(x + y)(\delta(S)) \geq 2, \tag{20}$$

is facet-inducing for 1-RPP(G) if subgraph $(S, E_{NR}(S))$ is 3-edge connected and either $V \backslash S = \{1\}$, or subgraph $(V \backslash S, E_{NR}(V \backslash S))$ is 3-edge connected.

Proof: Let $ax+by \geq c$, i.e., $\sum_{e\in E_R} a_e x_e + \sum_{e\in E_{NR}} a_e x_e + \sum_{e\in E_{NR}} b_e y_e \geq c$, be a valid inequality such that $\{(x,y) \in$ 1-RPP$(G) : \ (x+y)(\delta(S)) = 2\} \subseteq \{(x,y) \in$ 1-RPP$(G) : \ ax+by = c\}$. We have to prove that this inequality is a linear combination of the equalities (3) and $(x+y)(\delta(S)) \geq 2$. Note that this means to prove that

$$\begin{aligned} a_e = b_e &= 0, \ \ \forall e \in E_{NR}(S) \cup E_{NR}(V\setminus S), \\ a_e = b_e &= \alpha, \ \ \forall e \in \delta(S), \\ c &= \sum_{e\in E_R} a_e + 2\alpha. \end{aligned}$$

Consider the 1-RPP tour T formed with two copies of each edge in $E_{NR}(S) \cup E_{NR}(V\setminus S)$ and then replacing one copy of each $e \in E'_{NR}$ by the required edge parallel to e, plus two copies of a given edge $f \in \delta(S)$. Since this tour satisfies $(x+y)(\delta(S)) = 2$, it also satisfies $ax+by=c$, and we have

$$\sum_{e\in E_R} a_e + \sum_{e\in E_{NR}\setminus\delta(S)} a_e + \sum_{e\in E''_{NR}\setminus\delta(S)} b_e + a_f + b_f = c. \tag{21}$$

For each edge $e \in E''_{NR}\setminus\delta(S)$, consider the tour above except for $x_e = y_e = 0$ (it is a 1-RPP tour because subgraphs $G(S)$ and $G(V\setminus S)$ are 2-edge connected). By comparing both tours we obtain $a_e + b_e = 0$ for each edge $e \in E''_{NR}\setminus\delta(S)$.

For each edge $e \in E'_{NR}(S)$, given that $(S, E_{NR}(S))$ is 2-edge connected, the graph $(S, E_{NR}(S)\setminus\{e\})$ is connected and there is a T-join in it connecting its R-odd vertices that can be completed with edges in $E''_{NR}(S)$ used twice. A similar vector can be defined in the (connected) graph $(V\setminus S, E_{NR}(V\setminus S))$ and, after adding two copies of a given edge $f \in \delta(S)$ we have a 1-RPP tour T^* that does not traverses e and satisfies $(x+y)(\delta(S)) = 2$. The 1-RPP tour T^{*+2e} also satisfies $(x+y)(\delta(S)) = 2$ and, by subtracting the corresponding equations we obtain that $a_e + b_e = 0$. A similar argument for each edge $e \in E'_{NR}(V\setminus S)$ also leads to $a_e + b_e = 0$. Hence, we have $a_e + b_e = 0$ for each edge $e \in E_{NR}\setminus\delta(S)$.

For each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ either on $G(V\setminus S)$ or in $G(S)$ formed by non-required edges, the tour $T - \mathcal{C}$ obtained after removing from T one copy of each edge in $\mathcal{C}$ is also a 1-RPP tour satisfying $(x+y)(\delta(S)) = 2$. Proceeding as in the proof of Theorem 2 we obtain $a(\mathcal{C}') + b(\mathcal{C}'') = 0$.

Let $e = (i,j) \in E'_{NR}(V\setminus S)$ (if any). Since that $(V\setminus S, E_{NR}(V\setminus S))$ is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from e. Consider the three cycles $\mathcal{P}_1 \cup \{e\}$, $\mathcal{P}_2 \cup \{e\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$, for which equation $a(\mathcal{C}') + b(\mathcal{C}'') = 0$ holds. Hence, we have:
$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a_e = 0$, $a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + a_e = 0$, and $a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$, and we obtain $a_e = 0$. Furthermore, as $a_e + b_e = 0$ holds, we have $b_e = 0$ for all $e \in E'_{NR}(V\setminus S)$.

Let be $e = (i,j) \in E''_{NR}(V\setminus S)$ (if any). Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two edge-disjoint paths as above and consider the three cycles $\mathcal{P}_1 \cup \{e\}$, $\mathcal{P}_2 \cup \{e\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. Now we have:
$a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + b_e = 0$, $a(\mathcal{P}'_2) + b(\mathcal{P}''_2) + b_e = 0$, and $a(\mathcal{P}'_1) + b(\mathcal{P}''_1) + a(\mathcal{P}'_2) + b(\mathcal{P}''_2) = 0$, from which we obtain $b_e = 0$ and, as $a_e + b_e = 0$ holds, we have $a_e = 0$ for all $e \in E''_{NR}(V\setminus S)$.

Hence, $a_e = b_e = 0$ for all $e \in E_{NR}(V \setminus S)$. In a similar way, since $(S, E_{NR}(S))$ is a 3-connected graph, we obtain $a_e = b_e = 0$ also for each edge $e \in E_{NR}(S)$.

Let us denote the edges in $\delta(S)$ as $e_1, \ldots, e_p$, where $p \geq 3$ since graph G is 3-edge connected. The same argument used in Theorem 6 leads to prove that $a_{e_i} = a_{e_j} = b_{e_i} = b_{e_j}$ for all e_i, e_j.

By replacing $a_e = b_e = 0$ for each $e \in E_{NR} \setminus \delta(S)$, $a_e = b_e = \alpha$ for each $e \in \delta(S)$ in equation (21) we obtain that $\sum_{e \in E_R} a_e + 2\alpha = c$, and after replacing all the previous facts in $ax + by \geq c$ we obtain

$$\sum_{e \in E_R} a_e x_e + \alpha\Big((x+y)(\delta(S))\Big) \geq \sum_{e \in E_R} a_e + 2\alpha,$$

which is a linear combination of the equalities (3) and $(x+y)(\delta(S)) \geq 2$. ♦

In the remaining of the paper, we present several new families of valid inequalities for the 1-RPP: parity, p-connectivity and K-C inequalities.

3 Parity inequalities

In [7], the following constraints that generalize the well known co-circuit inequalities ([1]), were proposed for the MBCPP. They are called parity inequalities and, from Theorem 1, they are also valid for 1-RPP(G):

$$(x-y)(\delta(S)\setminus F) \geq (x-y)(F) - |F| + 1, \quad \forall S \subset V, \ \forall F \subseteq \delta(S) \text{ with } |F| \text{ odd.} \tag{22}$$

The above inequality can be simplified for the 1-RPP taking into account that, for each required edge e, we have $x_e = 1$ and there is no variable y_e. In general, either F and $\delta(S)\setminus F$ could contain required and non-required edges. However, it can be seen that the parity inequalities corresponding to sets where $\delta_R(S)\setminus F \neq \emptyset$ are not facet inducing. Hence, we will assume that $\delta(S)\setminus F \subset E_{NR}$. Let us denote here $F = F_R \cup F_{NR} = F_R \cup F'_{NR} \cup F''_{NR}$. By substituting $x_e = 1$ and deleting variables y_e for $e \in F_R$ in (22) we obtain

$$(x-y)(\delta(S)\setminus F) \geq x(F_R) + (x-y)(F_{NR}) - |F| + 1, \quad \Longrightarrow$$

$$(x-y)(\delta(S)\setminus F) \geq |F_R| + (x-y)(F_{NR}) - |F| + 1, \quad \Longrightarrow$$

$$(x-y)(\delta(S)\setminus F) \geq (x-y)(F_{NR}) - |F_{NR}| + 1 \tag{23}$$

This parity inequality (23) can be understood in the following way: the 1-RPP tours for which $(x-y)(F_{NR}) = |F_{NR}|$ (all the non required edges in F traversed once) holds, and given that all the edges in F_R are traversed once and $|F|$ is odd, they satisfy $(x-y)(\delta(S)\setminus F) \geq 1$. For the other 1-RPP tours, the inequality says nothing ($(x-y)(\delta(S)\setminus F) \geq 0$). These inequalities cut off (infeasible) solutions in which there is a cut-set with an odd number of edges traversed exactly once (these edges define the set F) and the other edges are traversed twice or none.

In the case $\delta(S)\setminus F = \emptyset$, and hence $F = \delta(S)$, the parity inequality (23) is:

$$(x-y)(F_{NR}) \leq |F_{NR}| - 1, \tag{24}$$

while in the case $F_{NR} = \emptyset$ the parity inequality (23) is:

$$(x-y)(\delta(S)\setminus F) \geq 1. \tag{25}$$

However, note that both sets, F_{NR} and $\delta(S) \setminus F$, cannot be empty simultaneously, because $F_{NR} \cup \delta(S) \setminus F = \delta_{NR}(S)$ and, as we assume graph (V, E_{NR}) is 3-edge connected, $|\delta_{NR}(S)| \geq 3$ holds.

Note also that, unlike other routing problems, if $|F| = 1$, that is, $F = \{e\}$, then inequality (23)

$$(x-y)(\delta(S)\backslash\{e\}) \geq x_e - y_e, \quad \text{or} \quad (x-y)(\delta(S)\backslash\{e\}) \geq 1,$$

if $e \in E_{NR}$ or $e \in E_R$, respectively, is not a connectivity inequality (18) or (20).

Note 2 Before proving if some parity inequalities (23) induce facets of 1-RPP(G), we will describe two types of 1-RPP tours satisfying them with equality. Recall that $|F|$ is odd. We are going to build 1-RPP tours traversing $\delta(S)$ a number $|F|+1$ or $|F|-1$ of times. Let us consider a cut-set $\delta(S)$ such that graphs $G(S)$ and $G(V \setminus S)$ are connected. We select an even number of (copies of) edges in $\delta(S)$ in the following two ways:

Type 1: If $\delta(S)\backslash F \neq \emptyset$, we select a copy of each edge in F and one more edge in $\delta(S)\backslash F$.

Type 2: If $F_{NR} \neq \emptyset$, we select one copy of each edge in F, except one edge in F_{NR}.

Note that, in both cases, we have selected an even number of copies of edges in $\delta(S)$. Let $V_o \subset V \setminus S$ be the set of vertices incident with an odd number of these selected edges. Given that the number of edges is even, $|V_o|$ is also even, and there is a T-join in $(V \setminus S, E_{NR}(V \setminus S))$. This same process is done in $(S, E_{NR}(S))$. Consider two copies of each non-required edge in $G(V \setminus S)$ and in $G(S)$ not belonging to the T-joins. Now, replace a copy of each edge in E'_{NR} by its corresponding parallel required edge. All these edges plus the two T-joins, plus the selected edges in $\delta(S)$, define a 1-RPP tour (it is even and connected and traverses all the required edges). It satisfies (23) with equality because

$(x-y)(\delta(S)\backslash F) = 1$ and $(x-y)(F_{NR}) = |F_{NR}|$ (*Type 1*), or

$(x-y)(\delta(S)\backslash F) = 0$ and $(x-y)(F_{NR}) = |F_{NR}| - 1$ (*Type 2*).

Theorem 8 *Parity inequalities (23), for all $S \subset V$, $F \subseteq \delta(S)$ with $|F|$ odd and $\delta_R(S) \subseteq F$ (and, hence, $\delta(S)\backslash F \subset E_{NR}$), are facet-inducing for 1-RPP(G) if subgraph $(S, E_{NR}(S))$ is 3-edge connected and either $V \backslash S = \{1\}$, or subgraph $(V \backslash S, E_{NR}(V \backslash S))$ is 3-edge connected.*

Proof: Let us denote here $F = F_R \cup F_{NR} = F_R \cup F'_{NR} \cup F''_{NR}$. Inequalities (23) can be written in the following way:

$$(x-y)(\delta(S)\backslash F) - (x-y)(F_{NR}) \geq 1 - |F_{NR}|. \tag{26}$$

Let us suppose there is another valid inequality $ax + by \geq c$,

$$\sum_{e \in E_R} a_e x_e + \sum_{e \in E_{NR}} a_e x_e + e \sum_{e \in E_{NR}} b_e y_e \geq c, \tag{27}$$

such that

$$\{(x,y) \in \text{1-RPP}(G) : \ (x-y)(\delta(S)\backslash F) - (x-y)(F_{NR}) = 1 - |F_{NR}|\} \subseteq$$
$$\subseteq \{(x,y) \in \text{1-RPP}(G) : \quad ax + by = c\}.$$

We have to prove that inequality (27) is a linear combination of equalities (3) and $(x - y)(\delta(S)\setminus F) - (x-y)(F_{NR}) \geq 1 - |F_{NR}|$.

Let $e \in E_{NR} \setminus \delta(S)$. Given that graphs $(S, E_{NR}(S))$ and $(V \setminus S, E_{NR}(V\setminus S))$ are 3-edge connected, they remain connected after deleting edge e, and there is a 1-RPP tour T in $G\setminus\{e\}$ that satisfies (26) with equality (see Note 2). The 1-RPP tour T^{+2e}, obtained from T by adding two traversals of edge e, also satisfies (26) with equality. By comparing the equations $ax + by = c$ corresponding to both tours, we obtain $a_e + b_e = 0 \quad \forall e \in E_{NR} \setminus \delta(S)$.

Let T be a 1-RPP tour T build as in Note 2 that traverses all the non-required edges in $G(V \setminus S)$ and $G(S)$. If any edge $e' \in E_{NR}$ is not traversed because it was in the T-join and has been replaced by its corresponding parallel edge $e \in E_R$, we add two copies of e' to T). For each cycle $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ either on $G(V\setminus S)$ or in $G(S)$ formed with non-required edges, the tour $T - \mathcal{C}$ obtained after removing from T one copy of each edge in $\mathcal{C}$ is also a 1-RPP tour satisfying (26) with equality. Proceeding as in the proof of Theorem 2 we obtain $a(\mathcal{C}') + b(\mathcal{C}'') = 0$.

Let $e = (i,j) \in E'_{NR}(V\setminus S)$ (if any). Since $(V\setminus S, E_{NR}(V\setminus S))$ is a 3-connected graph, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining vertices i and j with non-required edges different from e. Consider the three cycles $\mathcal{P}_1 \cup \{e\}$, $\mathcal{P}_2 \cup \{e\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$, for which equation $a(\mathcal{C}') + b(\mathcal{C}'') = 0$ holds. Hence, we have:

$$a(\mathcal{P}_1') + b(\mathcal{P}_1'') + a_e = 0, \;\; a(\mathcal{P}_2') + b(\mathcal{P}_2'') + a_e = 0, \text{ and } \;\; a(\mathcal{P}_1') + b(\mathcal{P}_1'') + a(\mathcal{P}_2') + b(\mathcal{P}_2'') = 0,$$

and we obtain $a_e = 0$. Furthermore, as $a_e + b_e = 0$ holds, we have $a_e = b_e = 0$ for all $e \in E'_{NR}(V\setminus S)$.

Let be $e = (i,j) \in E''_{NR}(V\setminus S)$ (if any). Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two edge-disjoint paths as above and consider the three cycles $\mathcal{P}_1 \cup \{e\}$, $\mathcal{P}_2 \cup \{e\}$, and $\mathcal{P}_1 \cup \mathcal{P}_2$. Now we have:

$$a(\mathcal{P}_1') + b(\mathcal{P}_1'') + b_e = 0, \;\; a(\mathcal{P}_2') + b(\mathcal{P}_2'') + b_e = 0, \text{ and } \;\; a(\mathcal{P}_1') + b(\mathcal{P}_1'') + a(\mathcal{P}_2') + b(\mathcal{P}_2'') = 0,$$

from which we obtain $b_e = 0$ and, as $a_e + b_e = 0$ holds, we have $a_e = b_e = 0$ for all $e \in E''_{NR}(V\setminus S)$.

In a similar way we obtain $a_e = b_e = 0$ for each edge $e \in E_{NR}(S)$. Hence, we have $a_e = b_e = 0$ for all $e \in E_{NR} \setminus \delta(S)$.

Let $e = (i,j) \in \delta_{NR}(S)$. If $e \in F_{NR}$, there is a 1-RPP tour of type 2 not traversing e. If $e \in \delta(S) \setminus F$ and $F_{NR} \neq \emptyset$ then there is a 1-RPP tour of type 2 not traversing e. If $e \in \delta(S) \setminus F$ and $F_{NR} = \emptyset$ then $|\delta(S)\setminus F| \geq 3$ and there is a 1-RPP tour ot type 1 not traversing e (traversing another edge in $\delta(S)\setminus F$). In any case, there is a 1-RPP tour T that satisfies (26) with equality and does not traverses e. The tour T^{+2e} also satisfies (26) with equality and, by comparing the equations $ax + by = c$ corresponding to both tours, we obtain $a_e + b_e = 0$ for all $e \in \delta_{NR}(S)$.

Let us suppose there are $e_1, e_2 \in F_{NR}$. Let T^1 be the 1-RPP tour of type 2 that traverses once all the edges in F except e_1 (see Note 2), and let T^2 be a similar tour corresponding to edge e_2. Both tours satisfy (26) with equality and, by comparing them, and considering that $a_e = b_e = 0$ for all edges $e \in E_{NR} \setminus \delta(S)$ we obtain $a_{e_1} = a_{e_2}$. By iterating this argument, we obtain $a_e = \lambda$ for all $e \in F_{NR}$. Furthermore, since $a_e + b_e = 0$ for each $e \in \delta_{NR}(S)$, we have $b_e = -\lambda$. Hence, $a_e = \lambda$ and $b_e = -\lambda$ for all $e \in F_{NR}$. Note that this is obviously true if F_{NR} contains only one edge.

Let us suppose there are $e_1, e_2 \in \delta(S) \setminus F$. Let T^1 be the 1-RPP tour of type 1 that traverses once the edges in $F \cup \{e_1\}$ (see Note 2), and let T^2 be the tour that traverses once the edges in $F \cup \{e_2\}$. Both tours satisfy (26) with equality and, by comparing them, and considering that $a_e = b_e = 0$ for all edges $e \in E_{NR} \setminus \delta(S)$ we obtain $a_{e_1} = a_{e_2}$. By iterating this argument, we obtain $a_e = \mu$ for all $e \in \delta(S) \setminus F$ and, hence, $b_e = -\mu$ for all $e \in \delta(S) \setminus F$. Again, this is obviously true if $\delta(S) \setminus F$ contains only one edge.

If both sets F_{NR} and $\delta(S) \setminus F$ are non-empty, let $e_1 \in F_{NR}$ and $e_2 \in \delta(S) \setminus F$. Let T^1 be the 1-RPP tour of type 1 that traverses once the edges in $F \cup \{e_2\}$, and T^2 the 1-RPP tour of type 2 that does not traverses e_1 nor e_2. Both tours satisfy (26) with equality and, by comparing them, we obtain $a_{e_1} + a_{e_2} = 0$ and, therefore, $\lambda = -\mu$. Hence, we have $a_e = \lambda$, $b_e = -\lambda$ for all $e \in F_{NR}$, and $a_e = -\lambda$, $b_e = \lambda$ for all $e \in \delta(S) \setminus F$.

By substituting all the previously computed coefficients a_e, b_e in inequality (27) we obtain

$$\sum_{e \in E_R} a_e x_e - \lambda\Big((x-y)(\delta(S) \setminus F) - (x-y)(F_{NR})\Big) \geq c.$$

Given that any of the 1-RPP tours T above satisfies this inequality with equality, we obtain

$$\sum_{e \in E_R} a_e - \lambda\Big(1 - |F_{NR}|\Big) = c$$

and, hence, inequality (27) reduces to

$$\sum_{e \in E_R} a_e x_e - \lambda\Big((x-y)(\delta(S) \setminus F) - (x-y)(F_{NR})\Big) \geq \sum_{e \in E_R} a_e - \lambda\Big(1 - |F_{NR}|\Big),$$

which is a linear combination of equalities (3) and $(x-y)(\delta(S) \setminus F) - (x-y)(F_{NR}) \geq 1 - |F_{NR}|$.
♦

Note 3 Theorem 8 also applies if one of the two shores S or $V \setminus S$ is formed only with one vertex.

Theorem 9 *¿The following is a complete formulation for the 1-RPP:?*

$$\left.\begin{array}{rl}
x_e = 1, & \forall e \in E_R \\
x_e \geq y_e, & \forall e \in E_{NR} \\
\text{Connectivity inequalities} & (18) + (20) \\
\text{Parity inequalities} & (23) \\
x_e \in \{0,1\}, & \forall e \in E \\
y_e \in \{0,1\}, & \forall e \in E_{NR}
\end{array}\right\} \quad (28)$$

Proof?: We have to prove that any solution (x^*, y^*) of (28) is a feasible solution for the 1-RPP. Obviously, (x^*, y^*) is a binary vector ($x_e^*, y_e^* \in \{0,1\}$, $\forall e \in E$) that represents a graph on the edges of G satisfying that the second copy of an edge e, $y_e^* = 1$, only exists if the first copy does ($x_e^* \geq y_e^*$), and that it contains each required edge ($x_e^* = 1$, $\forall e \in E_R$). It only remains to prove that the graph represented by (x^*, y^*) is an Eulerian graph (is connected and even).

Let us first suppose that the graph represented by (x^*, y^*) is not connected. Then, there exist a set $S \subset V \setminus \{1\}$ such that the cut-set $\delta(S)$ is not traversed, i.e., $(x^* + y^*)(\delta(S)) = 0$, while some edge in $E(S)$ is traversed, say $x^*_f = 1$ for any $f \in E(S)$. Then, the corresponding connectivity inequality (18) (if $f \in E_{NR}$) or (20) (if $f \in E_R$) is not satisfied by (x^*, y^*).

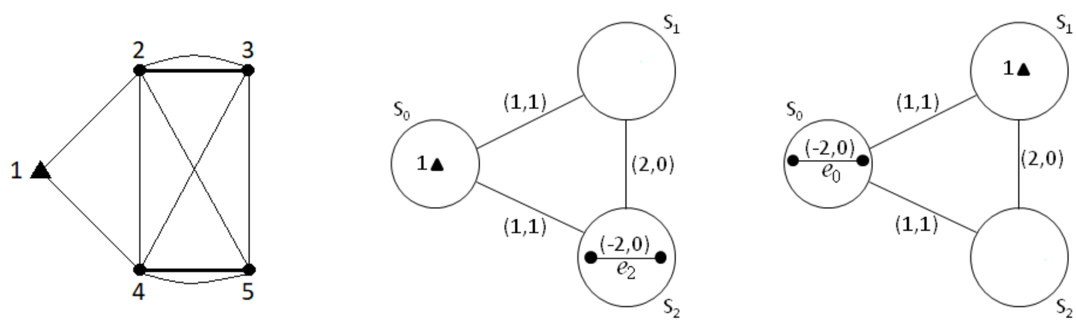
Let us now suppose that the graph represented by (x^*, y^*) is not an even graph. Then, there exist, at least, a cut-set $\delta(S)$ such that $(x^* + y^*)(\delta(S))$ is an odd number. Let $F \subseteq \delta(S)$ be the set of edges $e \in \delta(S)$ satisfying $x^*_e = 1$ and $y^*_e = 0$. Note that $|F|$ is odd. Note also that the edges in the set $\delta(S) \setminus F$ satisfy $x^*_e = y^*_e = 1$ or $x^*_e = y^*_e = 0$, i.e., $x^*_e - y^*_e = 0$. Then, if we substitute (x^*, y^*) in the Parity inequality (23) corresponding to $\delta(S)$ and F,

$$0 = (x^* - y^*)(\delta(S) \backslash F) \geq (x^* - y^*)(F_{NR}) - |F_{NR}| + 1 = 1$$

and this inequality is not satisfied by (x^*, y^*). ♦

4 p-connectivity inequalities

The constraints described in this section were introduced in [7] for the MBCPP to cut off fractional solutions as the one described as follows. Consider the 1-RPP instance shown in Figure 1(a), in which the depot is represented by a triangle, each thick line represents a required edge and each thin line represents a non-required one. Consider the fractional solution (x^*, y^*) with values $x^*_{(1,2)} = y^*_{(1,2)} = x^*_{(1,4)} = y^*_{(1,4)} = x^*_{(2,4)} = y^*_{(2,4)} = 0.5$, and $x^*_{(2,3)} = x^*_{(2,3)'} = x^*_{(4,5)} = x^*_{(4,5)'} = 1$, and the remaining variables equal to zero. It can be seen that this fractional solution satisfies all the inequalities presented in previous sections but it is cut off with the p-connectivity inequalities we present in what follows.



(a) (b) (c)

Figure 1: 2-connectivity inequalities.

Let $\{S_0, \ldots, S_p\}$ be a partition of V such that $\delta(S_i) \cap E_R = \emptyset$ for all i. Assume we divide the set $\{0, 1, \ldots, p\} = \mathcal{R} \cup \mathcal{N}$ (from 'Required' and 'Non-required') in such a way that

- $i \in \mathcal{R}$ if either $1 \in S_i$ or $E_R(S_i) \neq \emptyset$ (note that $1 \leq |\mathcal{R}| \leq p+1$), and
- $i \in \mathcal{N}$ if $1 \notin S_i$ and $E_R(S_i) = \emptyset$ (note that $0 \leq |\mathcal{N}| \leq p$, and $|\mathcal{R}| + |\mathcal{N}| = p + 1$),

and select one edge $e_i \in E(S_i)$ for every $i \in \mathcal{N}$. Note that $e_i \in E''_{NR}$. The following inequality

$$(x+y)(\delta(S_0)) + 2 \sum_{1 \le r < t \le p} x(S_r : S_t) \ge 2 \sum_{i \in \mathcal{N}} x_{e_i} + 2\,(|\mathcal{R}| - 1) \tag{29}$$

will be referred to as a p-connectivity inequality. Note that it is valid for the 1-RPP because the following p-connectivity inequalities

$$(x+y)(\delta(S_0)) + 2 \sum_{1 \le r < t \le p} x(S_r : S_t) \ge 2 \sum_{i=0, i \ne d}^{p} x_{e_i} \tag{30}$$

are valid for the MBCPP (it is assumed that $1 \in S_d$) and inequalities (29) are obtained from (30) after replacing the equalities $x_{e_i} = 1$ for all $i \in \mathcal{R}$ (required edges).

This inequality with $p = 2$ and $|\mathcal{N}| = 1$ is represented in Figures 1(b) and 1(c), where for each pair (a, b) associated with an edge e, a and b represent the coefficients of x_e and y_e, respectively.

Theorem 10 *p-connectivity inequalities (29) are facet-inducing for 1-RPP(G) if subgraphs $(S_i, E_{NR}(S_i))$, $i = 0, \ldots, p$, are 3-edge connected, $|(S_0 \colon S_i)| \ge 2$, $\forall\ i = 1, \ldots, p$, and the graph induced by $V \setminus S_0$ is connected.*

Proof: We will assume that $1 \in S_0$. The case $1 \in S_i$, $i \ne 0$, is similar and the proof is omitted here for the sake of brevity. Inequality (29) can be written as:

$$(x+y)(\delta(S_0)) + 2 \sum_{1 \le r < t \le p} x(S_r : S_t) - 2 \sum_{i \in \mathcal{N}} x_{e_i} \ge 2|\mathcal{R}| - 2. \tag{31}$$

Let us suppose there is another valid inequality $ax + by \ge c$,

$$\sum_{e \in E_R} a_e x_e + \sum_{e \in E_{NR}} a_e x_e + e \sum_{e \in E_{NR}} b_e y_e \ge c, \tag{32}$$

such that

$$\begin{aligned} \{(x,y) \in \text{1-RPP}(G) : \quad & (x+y)(\delta(S_0)) + 2 \sum_{1 \le r < t \le p} x(S_r : S_t) - 2 \sum_{i \in \mathcal{N}} x_{e_i} = 2|\mathcal{R}| - 2\} \subseteq \\ & \subseteq \{(x,y) \in \text{1-RPP}(G) : \quad ax + by = c\}. \end{aligned}$$

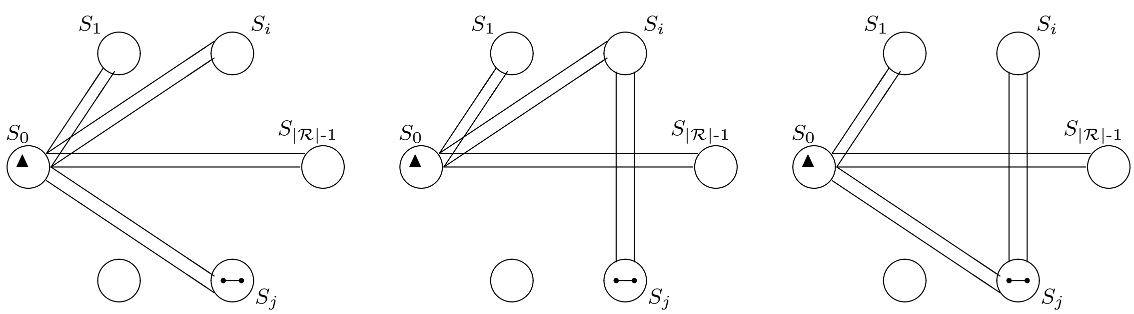
We have to prove that inequality (32) is a linear combination of the equalities (3) and inequality (31).

In the 1-RPP tours used in this proof we will not describe how the edges in each set $E(S_i)$ are traversed. It can be seen that all these tours can be completed by using T-joins, connecting with non-required edges traversed twice and replacing a traversal of each edge in E'_{NR} by the traversal of its parallel required edge, as described in Note 2 for the parity inequalities.

Similar arguments to those used in the proof of Theorem 8, lead to prove that $a_e + b_e = 0$, for each $e \in E_{NR}(S_i)$, $i \in \mathcal{R}$ and for each $e \in E_{NR}(S_i) \setminus \{e_i\}$, $i \in \mathcal{N}$. Furthermore, using the

3-edge connectivity of graph $(S_i, E_{NR}(S_i))$ (hence, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining the end-vertices of e with non-required edges different from e), we obtain that $b_e = 0$. Hence, we have $a_e = b_e = 0$ for all $e \in E_{NR}(S_i)$, $i \in \mathcal{R}$ and for all $e \in E_{NR}(S_i) \setminus \{e_i\}$, $i \in \mathcal{N}$.

Let S_i and S_j, $i, j \neq 0$ be two sets such that there is an edge $e \in (S_i : S_j)$. Note that $e \in E''_{NR}$. For the sake of simplicity, let us assume $i \in \mathcal{R}$, $j \in \mathcal{N}$ (with the other possibilities we would proceed similarly). Since all the sets $(S_0 : S_k)$ are non-empty, and subgraph $(S_j, E_{NR}(S_j))$ is 3-edge connected, we can construct the 1-RPP tour that traverses twice an edge $f \in (S_0 : S_j)$, traverses once the edge e_j, traverses all the required edges, and visits all the sets S_i, $i \in \mathcal{R}$ (see Figure 2(a), where we assume $\mathcal{R} = \{0, \dots, |\mathcal{R}| - 1\}$ and $\mathcal{N} = \{|\mathcal{R}|, \dots, p\}$). This tour satisfies inequality (31) as an equality. If we compare this tour with the one obtained after removing the two traversals of f and all the traversals of edges in $E(S_j)$, we obtain $a_f + b_f + a_{e_j} = 0$. We construct two more 1-RPP tours satisfying (31) with equality such as those depicted in Figure 2(b) and 2(c). By comparing (a) and (b), we obtain $a_{0j} + b_{0j} = a_{ij} + b_{ij} = -a_{e_j}$, and by comparing (a) and (c) we obtain $a_{0i} + b_{0i} = a_{ij} + b_{ij} = -a_{e_j}$, where a_{kl} (b_{kl}) represents the coefficient of the variable x (y) corresponding to any edge in $(S_k : S_l)$. Given that the graph induced by $V \setminus S_0$ is connected, we can iterate this argument to conclude that $a_e + b_e = 2\lambda$ for every edge $e \in (S_i : S_j)$ (including $(S_0 : S_i)$), and $a_{e_i} = -2\lambda$ for each e_i, $i \in \mathcal{N}$. Given that graph $(S_i, E_{NR}(S_i))$ is 3-edge connected and $b_e = 0$ for all edge $e \in E(S_i) \setminus \{e_i\}$, by comparing a 1-RPP tour traversing e_i twice and the tour obtained by replacing the second traversal of e_i by the traversal of a path joining its end-vertices, we obtain $b_{e_i} = 0$ for each e_i, $i \in \mathcal{N}$.



(a) (b) (c)

Figure 2: 1-RPP tours satisfying (31) with equality

For each $i \in \{1, 2, \dots, p\}$, let e_1, e_2 be two edges in $(S_0 : S_i)$ (recall that $|(S_0 : S_i)| \geq 2$ holds). We have already proved that $a_{e_1} + b_{e_1} = a_{e_2} + b_{e_2} = 2\lambda$. It can be seen that we can construct four 1-RPP tours satisfying inequality (31) as an equality as follows. One tour traverses e_1 once and does not traverses e_2. Another tour traverses e_2 once and does not traverses e_1. By comparing these tours we obtain $a_{e_1} = a_{e_2}$ and, hence, $b_{e_1} = b_{e_2}$. The third tour traverses both e_1 and e_2 once, and the fourth one traverses e_1 twice and does not traverses e_2. By comparing them, we obtain $a_{e_2} = b_{e_1}$ and, hence, also $a_{e_1} = b_{e_2}$, and $a_{e_1} = b_{e_1} = a_{e_2} = b_{e_2} = \lambda$. Hence, $a_e = b_e = \lambda$ for each edge $e \in (S_0 : S_i)$, $i = 1, \dots, p$.

As above, let S_i and S_j, $i, j \neq 0$ be two sets such that there is an edge $e = (u, v) \in (S_i : S_j)$ (again with $i \in \mathcal{R}$, $j \in \mathcal{N}$, for example). There is a 1-RPP tour T that traverses once edge e, an edge $e_i \in (S_0 : S_i)$, and an edge $e_j \in (S_0 : S_j)$ and satisfies inequality (31) as an equality. If we remove in T the traversal of e and add the traversal of the edges in a path joining u and v

formed with edges e_i, e_j plus some edges in $G(S_0)$, $G(S_i)$ and $G(S_j) \setminus \{e_j\}$ (if any of these last edges is traversed three times, two copies would be removed), we obtain another 1-RPP tour satisfying (31) as an equality. By comparing both tours we obtain $a_e = b_{e_i} + b_{e_j} = 2\lambda$, which implies $b_e = 0$ (recall that $a_e + b_e = 2\lambda$). Hence, $a_e = 2\lambda, b_e = 0$, for each edge $e \in (S_i : S_j)$, $i \neq j$.

By substituting all the previously computed coefficients a_e, b_e in inequality (32) we obtain

$$\sum_{e \in E_R} a_e x_e - 2\lambda \sum_{i \in \mathcal{N}} x_{e_i} + \lambda x(\delta(S_0)) + 2\lambda \sum_{1 \leq r < t \leq p} x(S_r : S_t) + \lambda y(\delta(S_0)) \geq c \implies$$

$$\sum_{e \in E_R} a_e x_e + \lambda (x + y)(\delta(S_0)) + 2\lambda \sum_{1 \leq r < t \leq p} x(S_r : S_t) - 2\lambda \sum_{i \in \mathcal{N}} x_{e_i} \geq c.$$

Given that the 1-RPP tour in Figure 2(a) after removing the two traversals of f and all the traversals in $G(S_j)$, for example, satisfies this inequality with equality, we obtain

$$\sum_{e \in E_R} a_e + 2\lambda \left(|\mathcal{R}| - 1\right) + 0 + 0 = c,$$

and, hence, inequality (32) reduces to

$$\sum_{e \in E_R} a_e x_e + \lambda (x + y)(\delta(S_0)) + 2\lambda \sum_{1 \leq r < t \leq p} x(S_r : S_t) - 2\lambda \sum_{i \in \mathcal{N}} x_{e_i} \geq \sum_{e \in E_R} a_e + \lambda \left(2|\mathcal{R}| - 2\right),$$

which is a linear combination of the equalities (3) and inequality (31). ♦

5 K-C inequalities

K-C inequalities were introduced and proved to be facet-inducing for the undirected Rural Postman Problem (RPP) in [9]. Since then, several variants of K-C inequalities have been proposed for many other arc routing problems. In this section we describe a new version of these inequalities and prove they are valid and facet-inducing for the 1-RPP. We keep calling them K-C inequalities for the sake of simplicity.

Consider the 1-RPP instance shown in Figure 3, in which the depot is represented by a triangle, each thick line represents a required edge, each thin line represents a non-required one, and each large circle represents an arbitrary subgraph containing at least a required edge.

Let (x^*, y^*) be the fractional solution with $x^*_e = 1$ for the required edges, $x^*_e = y^*_e = 0$ for the corresponding parallel non-required edges, $x^*_{(2,4)} = y^*_{(2,4)} = x^*_{(3,5)} = y^*_{(3,5)} = 0.5$ and $x^*_{(6,7)} = 1, y^*_{(6,7)} = 0$. This solution is "connected" but is not "even" at vertex 2 nor at vertex 3. Furthermore, it cannot be cut off with parity inequalities: For example, associated with the cut-set $\delta(\{2\})$ and $F = \{(1,2),(2,3),(2,4)\}$ we have the following parity inequality (23)

$$x_{(1,2)'} - y_{(1,2)'} + x_{(2,3)'} - y_{(2,3)'} \geq x_{(2,4)} - y_{(2,4)} - 1 + 1,$$

which is not violated by (x^*, y^*) (as $0 \geq 0$ holds). Note that the fractional solution similar to (x^*, y^*) except for $x^*_{(2,4)} = x^*_{(3,5)} = 1,\ y^*_{(2,4)} = y^*_{(3,5)} = 0$, is indeed cut off by the above parity

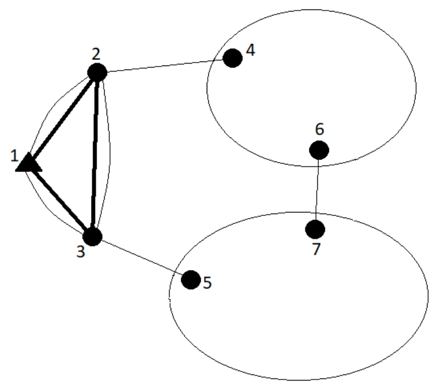


Figure 3: A 1-RPP instance where a K-C inequality appear.

inequality. It can also be seen that (x^*, y^*) satisfies all the p-connectivity inequalities (29). However, (x^*, y^*) is cut with the inequalities presented in this section.

Let $\{S_0, \ldots, S_K\}$, with $K \geq 3$, be a partition of V such that $\delta(S_i) \cap E_R = \emptyset$ for all $i = 1, 2, \ldots, K-1$. Assume we divide the set $\{1, \ldots, K-1\} = \mathcal{R} \cup \mathcal{N}$ (from 'Required' and 'Non-required') in such a way that

- $i \in \mathcal{R}$ if either $1 \in S_i$ or $E_R(S_i) \neq \emptyset$ (note that $0 \leq |\mathcal{R}| \leq K-1$), and
- $i \in \mathcal{N}$ if $1 \notin S_i$ and $E_R(S_i) = \emptyset$ (note that $0 \leq |\mathcal{N}| \leq K-1$, and $|\mathcal{R}| + |\mathcal{N}| = K-1$),

and select one edge $e_i \in E(S_i)$ for every $i \in \mathcal{N}$. Note that $e_i \in E''_{NR}$. Let $F \subseteq (S_0 : S_K)$ be a set of edges, with $|F| \geq 2$ and even, and $(S_0 : S_K)_R \subseteq F$. Let us denote here $F = F_R \cup F_{NR} = F_R \cup F'_{NR} \cup F''_{NR}$. The K-C inequalities for the 1-RPP are defined as:

$$(K-2)(x-y)\Big((S_0 : S_K) \setminus F\Big) - (K-2)(x-y)(F_{NR}) +$$
$$+ \sum_{\substack{0 \leq i < j \leq K \\ (i,j) \neq (0,K)}} \Big((j-i)x(S_i : S_j) + (2-j+i)y(S_i : S_j)\Big) - 2\sum_{i \in \mathcal{N}} x_{e_i} \geq 2|\mathcal{R}| - (K-2)|F_{NR}| \quad (33)$$

The coefficients and structure of the K-C inequalities are shown in Figure 4, where we assume $\mathcal{R} = \{1, \ldots, |\mathcal{R}|\}$ and $\mathcal{N} = \{|\mathcal{R}|+1, \ldots, K-1\}$. Edges in F (required and non-required, if any) are represented by thick lines. For each pair (a, b) associated with an edge e, a and b represent the coefficients of x_e and y_e, respectively.

K-C inequalities (33) are valid for 1-RPP(G) because they are obtained from the corresponding K-C inequality for the MBCPP after replacing each x_e by one and removing the y_e variables for all the required edges e. It is easy to see that, when $K = 2$, K-C inequality (33) reduces to a connectivity inequality (20) when $1 \in \mathcal{R}$ and to a connectivity inequality (18) when $1 \in \mathcal{N}$.

Regarding the fractional solution (x^*, y^*) described above for the instance represented in Figure 3, note that the K-C inequality (33) with $K = 3$, $F = \{(1,2), (2,3)\}$ and $(S_0 : S_3) \setminus F = \{(1,2)', (2,3)'\}$,

$$x_{(1,2)'} - y_{(1,2)'} + x_{(2,3)'} - y_{(2,3)'} + x_{(2,4)} + y_{(2,4)} + x_{(6,7)} + y_{(6,7)} + x_{(3,5)} + y_{(3,5)} \geq 4,$$

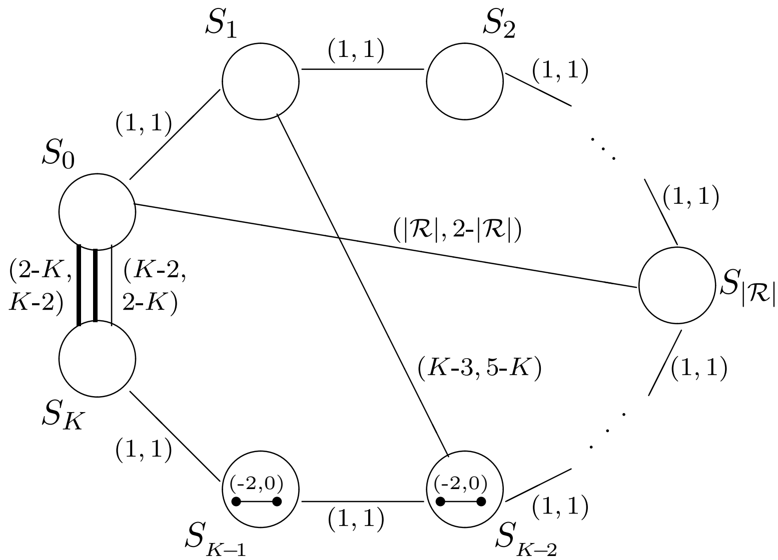


Figure 4: Coefficients of the K-C inequality

is violated by (x^*, y^*) (as $0 + 3 < 4$ holds).

Note 4 Les us describe several types of 1-RPP tours that satisfy the K-C inequality (33) with equality that will be used in the proof of Theorem 11. We do not detail how the edges in each set $E(S_i)$ are traversed. Note that if subgraphs $(S_i, E_{NR}(S_i))$, $i = 0, \ldots, K$, are 3-edge connected, all these tours can be completed by using T-joins as described in Note 2 for the parity inequalities. All of them traverse all the required edges. Note also that, although sets $(S_0 : S_K)\backslash F$ and F_{NR} could be empty sets, they cannot be empty simultaneously, because $F_{NR} \cup (S_0 : S_K)\backslash F = (S_0 : S_K)_{NR}$ and, as in Theorem 11 is assumed that $|F_R| \geq 2$, $(S_0 : S_K)$ contains at least two non-required edges.

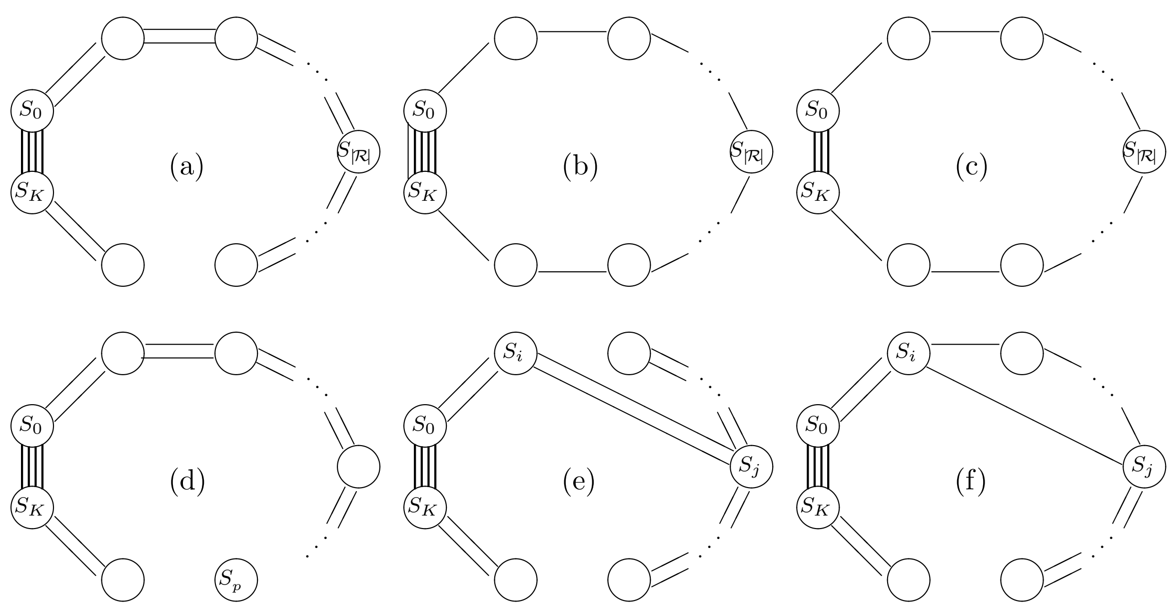


Figure 5: 1-RPP tours described in Note 4 and used in the proof of Theorem 11

(a) Tours traversing exactly once each edge in F, twice each edge e_i, for all $i \in \mathcal{N}$, and connecting sets S_j, $j = 0, 1, 2, \ldots, K-1$, with either two different edges in $(S_j : S_{j+1})$ used once or an edge used twice, as in Figure 5(a). Additionally, these tours could also traverse twice any edge (not drawn) in $(S_0 : S_K) \setminus F$. These tours satisfy (33) with equality:

$$-(K-2)|F_{NR}| + 2(K-1) - 2|\mathcal{N}| = 2|\mathcal{R}| - (K-2)|F_{NR}|.$$

(b) Tours traversing once each edge in F and one more edge in $(S_0 : S_K)$ (this could be a second traversal of an edge in F_{NR}), once each edge e_i, $i \in \mathcal{N}$, and connecting sets S_j, $j = 0, 1, 2, \ldots, K-1$, with exactly an edge in each set $(S_j : S_{j+1}), j = 0, \ldots, K-1$ (see Figure 5(b)). These tours satisfy (33) with equality:

$$(K-2) - (K-2)|F_{NR}| + K - 2|\mathcal{N}| = 2|\mathcal{R}| - (K-2)|F_{NR}|.$$

(c) Only if $F_{NR} \neq \emptyset$, tours traversing exactly once each edge in F except one of them in F_{NR}, once each edge e_i, $i \in \mathcal{N}$, and connecting sets S_j, $j = 0, 1, 2, \ldots, K-1$, with exactly an edge in each set $(S_j : S_{j+1}), j = 0, \ldots, K-1$ (see Figure 5(c)). These tours satisfy (33) with equality:

$$-(K-2)\Big(|F_{NR}| - 1\Big) + K - 2|\mathcal{N}| = 2|\mathcal{R}| - (K-2)|F_{NR}|.$$

(d) Tours traversing exactly once each edge in F, twice each edge e_i except one of them, say e_p, and connecting sets $S_j, j \neq p$ as in Figure 5(d). These tours satisfy (33) with equality:

$$-(K-2)|F_{NR}| + 2(K-2) - 2\Big(\|\mathcal{N}| - 1\Big) = 2|\mathcal{R}| - (K-2)|F_{NR}|.$$

(e) and (f) Tours traversing exactly once each edge in F, twice each edge e_i, for all $i \in \mathcal{N}$, and connecting sets S_j, $j = 0, 1, 2, \ldots, K-1$ as shown in Figure 5(e) and 5(f). These tours also satisfy (33) with equality.

Theorem 11 *K-C inequalities* (33) *are facet-inducing for 1-RPP(G) if subgraphs* $(S_i, E_{NR}(S_i))$, $i = 0, \ldots, K$, *are 3-edge connected,* $|(S_i : S_{i+1})| \geq 2$ *for* $i = 0, \ldots, K-1$, *and* $|F_R| \geq 2$.

Proof: Assume that $1 \in S_0$. The proof for the case $1 \in S_i$, $i \neq 0$, is similar. Let us suppose there is another valid inequality $ax + by \geq c$,

$$\sum_{e \in E_R} a_e x_e + \sum_{e \in E_{NR}} a_e x_e + \sum_{e \in E_{NR}} b_e y_e \geq c, \tag{34}$$

such that

$$\Big\{(x,y) \in \text{1-RPP}(G) : \ (x,y) \text{ satisfies (33) with equality}\Big\} \subseteq \Big\{(x,y) \in \text{1-RPP}(G) : \ ax + by = c\Big\}.$$

We have to prove that inequality (34) is a linear combination of the equalities (3) and inequality (33).

Let $e \in E_{NR}(S_i)$, $i \in \{0, 1, \ldots, K\}$, different from e_i if $i \in \mathcal{N}$. Similar arguments to those used in the proof of Theorem 8 using the 3-edge connectivity of graph $(S_i, E_{NR}(S_i))$ lead to prove that $a_e = b_e = 0$.

Let $e \in (S_0 : S_K)_{NR}$ and let T be a 1-RPP tour of type (c) in Note 4 that does not traverse edge e. The 1-RPP tour T^{+2e} also satisfies (33) with equality, since $x_e = y_e = 1$ and the sum of the coefficients of both variables in (33) is zero. By comparing the equations $ax + by = c$ corresponding to both tours, we obtain that $a_e + b_e = 0$, for all $e \in (S_0 : S_K)_{NR}$.

For each $i \in \mathcal{N}$, let T^1 be the 1-RPP tour of type (a) in Note 4 traversing twice an edge in each set $(S_j : S_{j+1})$, $j \neq i$ and let T^2 be the 1-RPP tour of type (d) traversing twice the same edge in each set $(S_j : S_{j+1})$, $j \neq i-1, i$. By comparing the corresponding equations $ax+by = c$ of both tours, we obtain that $a_e + b_e + a_{e_i} + b_{e_i} = 0$ for all $e \in (S_{i-1} : S_i)$. If we consider the 1-RPP tour T^3 of type (a) traversing twice an edge in each set $(S_j : S_{j+1})$, $j \neq i-1$, by comparing the equations corresponding to T^2 and T^3 we conclude $a_e + b_e + a_{e_i} + b_{e_i} = 0$ for all $e \in (S_i : S_{i+1})$. For each $i \in \mathcal{R}$, let T^1 and T^3 two 1-RPP tours of type (a) defined as above. By comparing them we conclude that $a_e + b_e = a_f + b_f$ for all $e \in (S_{i-1} : S_i)$ and $f \in (S_i : S_{i+1})$. By iterating this argument, we obtain that $a_e + b_e = 2\lambda$ for all $e \in (S_i : S_{i+1})$, $i = 1, \ldots, K-1$, and $a_{e_i} + b_{e_i} = -2\lambda$ for all $i \in \mathcal{N}$, where λ is a certain constant value.

For each $i \in \mathcal{N}$, let T^1 be the 1-RPP tour of type (b) in Note 4 traversing edge $e_i = (u, v)$ once. Given that $(S_i, E_{NR}(S_i))$ is 3-edge connected graph, we can find a path connecting u and v that does not use e_i. If we add this path plus one copy of e_i to T^1, we obtain a 1-RPP tour T^2 also satisfying (33) with equality. By comparing both tours, and given that $a_e = b_e = 0$ for all $e \in E(S_i) \setminus \{e_i\}$, we obtain $b_{e_i} = 0$ and, therefore, $a_{e_i} = -2\lambda$.

For each $i \in \{0, 1, 2, \ldots, K-1\}$, let e, f be two edges in $E(S_i : S_{i+1})$ (recall that $|(S_i, S_{i+1})| \geq 2$ holds). There are two 1-RPP tours T^1 and T^2 of type (b) in note 4 traversing edges e and f once respectively. Comparing both tours, we get $a_e = a_f$. Since we have proved that $a_e + b_e = 2\lambda = a_f + b_f$, we have $b_e = b_f$. Furthermore, let T^3 be a tour of type (a) traversing edge e twice and T^4 a similar tour traversing e and f once. By comparing these tours, we obtain $b_e = a_f$ and, since $a_f = a_e$, we get $a_e = b_e$. Therefore $a_e = b_e = \lambda$ for each edge $e \in E(S_i : S_{i+1})$, for all $i \in \{0, 1, 2, \ldots, K-1\}$.

Let $e \in F_{NR}$ (if any). By comparing the 1-RPP tour of type (b) traversing once all the edges in F except edge e that is traversed twice, and the 1-RPP tour of type (c) traversing once all the edges in F except edge e that is not traversed, we obtain that $a_e + b_e = 0$. On the other hand, by comparing the 1-RPP tour of type (a) traversing once all the edges in F and the previous 1-RPP tour of type (c) we obtain that $a_e + \lambda(K-1) - \lambda = 0$. Hence, $a_e = -\lambda(K-2)$ and $b_e = \lambda(K-2)$.

Let $e \in E(S_0 : S_K) \backslash F$ (if any). By comparing the 1-RPP tour of type (a) traversing once all the edges in F and not traversing e and the same tour by adding two copies of e we obtain that $a_e + b_e = 0$. On the other hand, by comparing the 1-RPP tour of type (a) traversing once all the edges in F and the 1-RPP tour of type (b) traversing once all the edges in $F \cup \{e\}$ we obtain that $-a_e + \lambda(K-1) - \lambda = 0$ and, hence, $a_e = -\lambda(K-2)$.

Finally, for each edge $e \in E(S_i : S_j)$, $|i-j| > 1$, comparing tours of type (a) and (e) in Figure 5, we obtain $a_e + b_e = 2\lambda$. Then, comparing tours of type (e) and (f), we obtain $b_e + \lambda\big(|i-j| - 1\big) = \lambda$. Therefore, $b_e = \lambda\big(2 - |i-j|\big)$ and $a_e = \lambda|i-j|$.

By substituting all the previously computed coefficients a_e, b_e in inequality (34) we obtain

$$\sum_{e\in E_R} a_e x_e + \lambda(K-2)(x-y)\Big((S_0:S_K)\setminus F\Big) - \lambda(K-2)(x-y)(F_{NR}) +$$
$$+\lambda \sum_{\substack{0\le i<j\le K \\ (i,j)\neq(0,K)}} \Big((j-i)x(S_i:S_j) + (2-j+i)y(S_i:S_j)\Big) - 2\lambda \sum_{i\in\mathcal{N}} x_{e_i} \ge c$$

Given that the 1-RPP tour of type (a) in Note 4, for example, satisfies this inequality with equality, we obtain

$$\sum_{e\in E_R} a_e - (K-2)|F_{NR}| + 2\lambda(K-1) - 2\lambda|\mathcal{N}| = c \quad \Rightarrow \quad \sum_{e\in E_R} a_e + \lambda\Big(2|\mathcal{R}| - (K-2)|F_{NR}|\Big) = c,$$

and, hence, inequality (34) is a linear combination of equalities (3) and inequality (33). ♦

6 Conclusions

We have studied the Rural Postman Problem (RPP) with two special features. First, it is defined in a graph with a non-required edge parallel to each required one. Second, it is formulated with three binary variables associated with the traversal of a required edge and its parallel non-required one, although some variables are superfluous. This model is interesting by itself and moreover it is the special case for $K = 1$ of the RPP with K vehicles (K-RPP). The polyhedron defined by the convex hull of the set of feasible solutions of the 1-RPP has been studied, proving that several wide families of inequalities are facet inducing of it. These results are used in [3] for the corresponding polyhedral study of the K-RPP with $K > 1$.

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